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On the parametrization of equilibrium stress fields in the Earth

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SUMMARY

A new method for parametrizing the possible equilibrium stress fields of a laterally heterogeneous earth model is described. In this method a solution of the equilibrium equations is first found that satisfies some desirable physical property. For example, we show that the equilibrium stress field with smallest norm relative to a given inner product can be obtained by solving a static linear elastic boundary value problem. We also show that the equilibrium stress field whose deviatoric component has smallest norm with respect to a given inner product can be obtained by solving a steady-state incompressible viscous flow problem. Having found such a solution of the equilibrium equations, all other solutions can be written as the sum of this equilibrium stress field and a divergence-free stress tensor field whose boundary tractions vanish. Given *n* divergence-free and traction-free tensor fields, we then obtain a simple *n*-dimensional parametrization of equilibrium stress fields in the earth model. The practical construction of such divergence- and traction-free tensor fields in the mantle of a spherically symmetric reference earth model is described using generalized spherical harmonics.

Key words: Inverse theory; Seismic tomography; Theoretical seismology.

1 INTRODUCTION

In this paper, we consider the problem of parametrizing the possible equilibrium stress fields in the Earth. These equilibrium stress fields are solutions of the equations of static equilibrium—or 'equilibrium equations' for short—which express a balance between the forces of self-gravitation, rotation, and internal stresses in the Earth (e.g. Dahlen & Tromp 1998, section 3.1). Implicit in the use of the equilibrium equations is the assumption that the dynamic component of any velocity field in the Earth associated with long-term geodynamic processes such as mantle convection is negligible. This assumption is justified for seismological applications by scaling analysis of the time-dependent momentum equations, which indicate that the Earth must be extremely close to a state of static equilibrium (e.g. Forte 2007, section 1.23.2.3.1).

The seismological interest in the equilibrium stress field is due to its occurrence as a parameter in the elastodynamic equations that govern seismic wave propagation (e.g. Rayleigh 1906; Love 1911; Dahlen 1972a, 1973; Dahlen & Smith 1975; Woodhouse & Dahlen 1978; Valette 1986; Vermeersen & Vlaar 1991; Dahlen & Tromp 1998). Consequently, it may be possible to make inferences about the equilibrium stress field from seismic observations. The feasibility of such an inverse problem depends upon the sensitivity of seismic observations to variations in the equilibrium stress field relative to their sensitivity to variations in other model parameters such as seismic wave speeds, density, anisotropy, anelasticity and boundary topography. This issue has previously been investigated by a number of authors including Dahlen (1972b,c) and Nikitin & Chesnokov (1984) who considered the effects of deviatoric equilibrium stress fields on body wave radiation patterns and traveltimes. Using estimates of the magnitude of deviatoric stresses in the Earth, Dahlen concluded that the influence of the equilibrium stress field on these body wave observations is likely small compared to the effects of other factors such as lateral variations in seismic wave speed. It is not, however, immediately clear that these conclusions also apply to longer period seismic observations such as normal mode spectra. For example, Valette (1986) determined an expression for the first-order perturbation to normal mode eigenfrequencies due to a deviatoric equilibrium stress field using the isolated mode approximation, and from examination of the resulting sensitivity kernel, concluded that there is no *a priori* reason to neglect the effects of deviatoric equilibrium stress fields.

Determining the effects of the equilibrium stress field on seismic wave propagation is complicated by the fact the perturbations to the density structure, to boundary topography, and to the equilibrium stress field cannot be made independently. This is because any perturbations to these parameters must be such that the equilibrium equations are satisfied in the perturbed earth model. As a result, the construction of a range of equilibrium stress fields for a given laterally heterogeneous earth model (i.e. one in which the density and boundary topography perturbations have been specified) is a non-trivial problem. Similarly, if we wish to include the equilibrium stress field as an unknown in a tomographic inversion, we require a method for parametrizing the perturbations to the equilibrium stress field consistent with given perturbations to the density structure and boundary topography.

The problem of parametrizing the possible equilibrium stress fields of an earth model has been considered previously by Backus

(1967) whose method has, for example, been applied practically by Dahlen (1982) and Valette & Chambat (2004). Backus's method makes use of a representation theorem for symmetric second order tensor fields in terms of six scalar potential functions (Backus 1966), and is based on the observation that the equilibrium equations place only three constraints on these six scalar potential functions. It follows that we can specify three of these scalar potential functions arbitrarily (subject to certain compatibility conditions), and then solve the equilibrium equations for the remaining three scalar potential functions. In this way we see that a unique equilibrium stress field corresponds to each possible choice the three arbitrary scalar potential functions.

In this paper, we approach the problem in a different manner. Our starting point is the observation that the difference between any two equilibrium stress fields is a divergence-free stress field whose boundary tractions vanish; the vector space of such stress fields is denoted by $\ker(\mathrm{Div}_0)$, this notation being fully explained in section 3.1. It follows that if, by some means, we have obtained a particular equilibrium stress field $\mathbf{T}_m$, then all other equilibrium stress fields can be written in the form $\mathbf{T}_m + \mathbf{S}$, where $\mathbf{S}$ is an element of $\ker(\mathrm{Div}_0)$. Consequently, given such an equilibrium stress field $\mathbf{T}_m$, along with a finite set $\{\mathbf{S}_i\}_{i=1}^n$ of elements of $\ker(\mathrm{Div}_0)$, we can then consider the expression

$$\mathbf{T} = \mathbf{T}_m + a_1\mathbf{S}_1 + \cdots + a_n\mathbf{S}_n, \quad (1)$$

where $a_i, \ldots, a_n$ are scalar constants, as forming an n-dimensional parametrization of equilibrium stress fields in the earth model (more formally, we can regard this expression as defining an affine mapping from $\mathbb{R}^n$ into the space of equilibrium stress fields). It is clear that by including a sufficiently large set of elements of $\ker(\mathrm{Div}_0)$ in the above expression we can, in principle, express any equilibrium stress field in this form.

Using this approach we have separated the problem of parametrizing the possible equilibrium stress fields into two sub-problems: (i) determining a particular solution $\mathbf{T}_m$ of the equilibrium equations and (ii) constructing a suitably large number of elements of $\ker(\mathrm{Div}_0)$. In solving the first of these problems we are free to seek a solution of the equilibrium equations that satisfies some physically desirable property. For example, we can attempt to find the equilibrium stress field with smallest norm with respect to a given inner product. Using an orthogonal decomposition theorem for second order symmetric tensor fields (e.g. Berger & Ebin 1969; Ting 1977; Georgescu 1980; Cantor 1981) we show that there is a unique solution to this problem, and that this equilibrium stress field—which we call the 'minimum equilibrium stress field' for the earth model—can be constructed by solving a boundary value problem that has exactly the same form as a static linear elastic displacement problem. Alternatively, we can seek the equilibrium stress field whose deviatoric component has the smallest norm with respect to the given inner product. This problem can be solved using the method of Lagrange multipliers, and we show that the resulting equilibrium stress field—which we call the 'minimum deviatoric equilibrium stress field'—can be obtained by solving a boundary value problem of the same form as the steady-state incompressible Navier–Stokes equations. The idea of determining the equilibrium stress field with the minimum deviatoric components has previously been considered by Dahlen (1981, 1982) in studies of isostasy in the oceanic lithosphere. However, Dahlen's approach to this problem differs from ours in a number of ways. First, his method is less general due to his use of a number assumptions about the form of the equilibrium stress field derived from consideration of local isostasy. Second, he adopts a 'local definition' of the 'minimum deviatoric equilibrium stress field' (for example, see eq. 21 in Dahlen 1981 or eq. 47 in Dahlen 1982) in contrast to our 'global definition' in terms of an inner product on the space of second order symmetric tensor fields. Because of these differences Dahlen was not led to the interesting relationship between the minimum deviatoric equilibrium stress field and the steady-state incompressible Navier–Stokes equations described in this work. In generating elements of the vector space $\ker(\mathrm{Div}_0)$ for use in eq. (1) there are a number of available methods (e.g. Truesdell 1959; Gurtin 1963). However, because we need only consider the construction of such tensor fields in a spherically symmetric reference model it is simplest to use Backus's method specialized to the case of divergence-free tensor fields. In doing this we do not use the scalar representation theorem of Backus (1966), but instead employ the generalized spherical harmonic formalism of Phinney & Burridge (1973) which, we feel, is more suited to practical calculations.

The method for parametrizing equilibrium stress fields described above provides an alternative to that given by Backus (1967). It will be useful to now briefly consider some of the merits of these two approaches. A disadvantage of our method is that the calculation of either the 'minimum equilibrium stress field' or the 'minimum deviatoric equilibrium stress field' requires the solution of a system of linear partial differential equations. This is in contrast to Backus's method which involves largely algebraic calculations. Consequently, the practical implementation of Backus's parametrization is simpler than ours. However, as is discussed in more detail in Section 5, this disadvantage is not very severe because the calculations involved in producing either minimum equilibrium stress field or the minimum deviatoric equilibrium stress field can be performed efficiently using a range of existing numerical techniques.

To illustrate a potential advantage of our method it will be useful to consider the problem of estimating the likely effects of deviatoric equilibrium stress fields on seismic wave propagation. To do this we must be able to produce a number of realistic equilibrium stress fields for a given earth model in which seismic calculations can be performed. Stating here precisely what is meant by 'realistic' is difficult because the state of stress within the Earth's interior is not well understood. Physical arguments and information derived from geodynamic simulations do, however, suggest some general properties of a realistic equilibrium stress field (e.g. Karato & Wu 1993). For example, it is reasonable to expect that the deviatoric component of the equilibrium stress field should be small relative to its hydrostatic component. This is because rocks in the Earth's interior would be expected undergo some form of deformation (e.g. fracture or plastic flow) if the deviatoric stresses became too large, and any such deformation would in turn act to lower the magnitude of the deviatoric stress. Using Backus's method it is not immediately clear how we could (other than by trial-and-error) determine the equilibrium stress field in the earth model having deviatoric components that are as small as possible. However, the 'minimum deviatoric equilibrium stress field' for an earth model provides an immediate solution to this problem.

As a further example of a potential advantage, suppose that we wished to perform a tomographic inversion in which density and boundary topography are free-parameters, but do not wish to include the equilibrium stress field as a parameter in the inversion. In such cases it has been usual to employ the so-called 'quasi-hydrostatic approximation' in which the deviation of the equilibrium stress field away from its hydrostatic reference value is neglected; clearly this approximation results in a perturbed earth model that will not be in static equilibrium (e.g. Dahlen & Tromp 1998, section 3.11). An alternative to using the quasi-hydrostatic approximation would be to set the equilibrium stress field during the tomographic inversion equal to either the minimum equilibrium stress field or the minimum deviatoric equilibrium stress field. Although in doing this we would be making an essentially arbitrary assumption, this process would lead to a self-consistent model parametrization in which the equilibrium stress fields used was physically plausible. More generally, if we wished to include the equilibrium stress field as a parameter in a tomographic inversion, then either the 'minimum equilibrium stress field' or the 'minimum deviatoric equilibrium stress field' would provide a sensible *a priori* equilibrium stress field about which to make small perturbations at each step of the tomographic inversion.

2 A REVIEW OF EQUILIBRIUM EQUATIONS

2.1 Statement of the basic equations

The earth model is supposed to occupy a compact subset $V \subseteq \mathbb{R}^3$ with smooth boundary ∂V, and is further divided into a number of solid and fluid subregions which are separated by smooth, non-intersecting, closed surfaces called internal boundaries. The union of the solid regions will be denoted V_S and that of all fluid regions V_F. The union Σ of all internal and external boundaries is split into the four subsets Σ_{SS}, Σ_{SF}, Σ_{FS} and Σ_{FF} where the first subscript denotes whether the region on the inside of the boundary is solid (S) or fluid (F), while the second subscript specifies whether the region on the outside of boundary is solid or fluid. We note that if the earth model has an ocean, then its free-surface is regarded, by definition, as a fluid–fluid boundary. A generic point in the earth model will be denoted $\mathbf{x}$ in what follows. The equilibrium stress tensor in the model will be written $\mathbf{T}$, or in component form T_{ji}. This stress tensor is symmetric and satisfies the equilibrium equations which may be written as

$$\mathrm{Div}\mathbf{T} = \rho \nabla \phi. \tag{2}$$

In this equations Div is the divergence operator on second order tensor fields whose action on $\mathbf{T}$ may be written in index notation as

$$(\mathrm{Div}\mathbf{T})_i = T_{ji,j}, \tag{3}$$

where we have made use of the summation convention, the 'comma' notation for partial derivatives, and have written Div for the divergence operator on second order tensor fields to distinguish it from the divergence operator on vector fields which we write as div. The boundary conditions on $\mathbf{T}$ are that the traction vector be continuous on all boundaries so that

$$[\hat{\mathbf{n}} \cdot \mathbf{T}]_-^+ = 0, \quad \mathbf{x} \in \Sigma, \tag{4}$$

where $\hat{\mathbf{n}}$ is the outward unit normal vector to a boundary, and the notation $[\cdot]_-^+$ denotes the jump in a quantity on crossing a boundary in the direction of the outward normal. The other terms in eq. (2) are the density ρ, and the gravitational potential ϕ which is a solution of the equation

$$\nabla^2 \phi = \begin{cases} 4\pi G \rho & \mathbf{x} \in V \\ 0 & \mathbf{x} \in \mathbb{R}^3 \backslash V \end{cases}, \tag{5}$$

where G is the universal gravitational constant, \ denotes the difference of two sets and where ϕ is subject to the boundary conditions

$$[\phi]_-^+ = 0, \quad \mathbf{x} \in \Sigma, \tag{6}$$

$$[\hat{\mathbf{n}} \cdot \nabla \phi]_-^+ = 0, \quad \mathbf{x} \in \Sigma, \tag{7}$$

along with the condition that ϕ vanish at infinity.

In a spherically symmetric earth model with radius b it may be shown that

$$\phi(r) = -4\pi G \left\{ \int_r^b \rho(s) s \, \mathrm{d}s + \frac{1}{r} \int_0^r \rho(s) s^2 \, \mathrm{d}s \right\}, \tag{8}$$

and that a solution of the equilibrium equations exists in the form $\mathbf{T} = -p\mathbf{1}$ with the pressure given by

$$p(r) = \int_r^b \rho(s) g(s) \, \mathrm{d}s, \tag{9}$$

where $g = \partial_r \phi$ is the gravitational acceleration in the earth model. A solution of the equilibrium equations taking the form $\mathbf{T} = -p\mathbf{1}$ is said to be 'hydrostatic'. It may be shown that in a hydrostatic earth model the level surfaces of the three scalar fields ρ, ϕ and p must all coincide, and that (in the absence of rotation) each such level surface must be spherical (e.g. Dahlen & Tromp 1998, section 13.11.1). Because of this constraint it is not possible to find an everywhere hydrostatic solution to the equilibrium equations in an earth model with a laterally heterogeneous density structure. In the fluid regions of a model, however, it is necessary for the stress tensor to be hydrostatic because a stationary fluid cannot support deviatoric stresses. We see shortly that this condition in fluid regions places a strong constraint on the possible density structures of laterally heterogeneous earth models. For simplicity, the effects of rotation have not been included in the above discussion. However, rotational effects can be incorporated into the hydrostatic reference model described below using the theory of hydrostatic ellipticity (Jeffreys 1976; Dahlen & Tromp 1998, section 14.1).

2.2 Linearized equations in a slightly laterally heterogeneous earth model

Let us now consider an earth model that is obtained from a spherically symmetric reference model with a hydrostatic equilibrium stress field by adding small perturbations to the density and boundary positions. The density in the perturbed model will be written

$$\rho = \rho^{(0)} + \rho^{(1)}, \tag{10}$$

where $\rho^{(0)}$ is the density in the spherically symmetric reference model and $\rho^{(1)}$ is the density perturbation; in what follows the superscripts (0) and (1) will be used to distinguish between quantities in the reference model their first-order perturbations. Each of the spherical boundaries in the reference model is deformed so that a point with spherical polar coordinates (r, θ, φ) on the reference boundary is moved to the point $(r + h(\theta, \varphi), \theta, \varphi)$ on the perturbed boundary. We may assume without loss of generality that the density and boundary perturbations are such that their average over any spherical surface vanishes, so, for example, the spherical average

$$\bar{\rho}^{(1)}(r) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \rho^{(1)}(r, \theta, \phi) \sin(\theta) \, \mathrm{d}\theta \, \mathrm{d}\varphi \tag{11}$$

of the density perturbation over a spherical surface with radius r is equal to zero; this condition is equivalent to requiring that the degree-zero spherical harmonic expansion coefficients of $\rho^{(1)}$ and h are equal to zero. The stress tensor in the perturbed model takes the form

$$\mathbf{T} = -p^{(0)}\mathbf{1} + \mathbf{T}^{(1)}, \tag{12}$$

in solid regions, and

$$\mathbf{T} = -p^{(0)}\mathbf{1} - p^{(1)}\mathbf{1}, \tag{13}$$

in fluid regions. Upon cancelling out the zeroth-order terms and ignoring any products of the perturbed quantities, the equilibrium equations in the perturbed model become

$$\mathrm{Div}\mathbf{T}^{(1)} = \rho^{(0)}\nabla\phi^{(1)} + \rho^{(1)}\nabla\phi^{(0)}, \tag{14}$$

in solid regions, and

$$-\nabla p^{(1)} = \rho^{(0)}\nabla\phi^{(1)} + \rho^{(1)}\nabla\phi^{(0)}, \tag{15}$$

in fluid regions. Linearizing the continuity of traction condition in the perturbed earth model leads to the following boundary conditions for $\mathbf{T}^{(1)}$ and $p^{(1)}$ which are applied on the unperturbed boundaries of the reference earth model

$$[\hat{\mathbf{n}}\cdot\mathbf{T}^{(1)} + \rho^{(0)}gh\hat{\mathbf{n}}]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma_{SS}, \tag{16}$$

$$\hat{\mathbf{n}}\cdot\mathbf{T}^{(1)} + p^{(1)}\hat{\mathbf{n}} - [\rho^{(0)}]_{-}^{+}gh\hat{\mathbf{n}} = 0, \quad \mathbf{x}\in\Sigma_{SF}, \tag{17}$$

$$\hat{\mathbf{n}}\cdot\mathbf{T}^{(1)} + p^{(1)}\hat{\mathbf{n}} + [\rho^{(0)}]_{-}^{+}gh\hat{\mathbf{n}} = 0, \quad \mathbf{x}\in\Sigma_{FS}, \tag{18}$$

$$[p^{(1)} - \rho^{(0)}gh]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma_{FF}, \tag{19}$$

where $g = \partial_r\phi^{(0)}$ (e.g. Dahlen & Tromp 1998, section 13.7.1). The perturbed gravitational potential $\phi^{(1)}$ in the above equations is a solution of the equation

$$\nabla^2\phi^{(1)} = \begin{cases} 4\pi G\rho^{(1)} & \mathbf{x}\in V \\ 0 & \mathbf{x}\in\mathbb{R}^3\setminus V \end{cases}, \tag{20}$$

subject to the linearized boundary conditions

$$[\phi^{(1)}]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma, \tag{21}$$

$$[\hat{\mathbf{n}}\cdot\nabla\phi^{(1)} + 4\pi G\rho^{(0)}h]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma, \tag{22}$$

along with the requirement that $\phi^{(1)}$ vanish at infinity. By considering the degree-zero spherical harmonic component of the above equation for $\phi^{(1)}$ it is clear that because the degree-zero components of $\rho^{(1)}$ and h both vanish the same is true of $\phi^{(1)}$. Similarly, we see from eq. (15) that the spherically averaged part $\bar{p}^{(1)}$ of the pressure perturbation $p^{(1)}$ is constrained to be constant in each connected component of V_F. The remaining aspherical part of the pressure perturbation is defined by

$$\hat{p}^{(1)} = p^{(1)} - \bar{p}^{(1)}, \tag{23}$$

and is seen to satisfy exactly the same equations and boundary conditions as $p^{(1)}$.

2.3 Constraints on the model perturbations due to the hydrostatic condition in fluid regions

We now consider in detail how the hydrostatic condition on the stress tensor in fluid regions constrains the possible model perturbations; the results of this section have been previously described by Backus (1967), Dahlen (1974), Woodhouse & Dahlen (1978) and Wahr & de Vries (1989). It will be useful to write the density perturbation as

$$\rho^{(1)} = \rho^{(1,S)} + \rho^{(1,F)}, \tag{24}$$

where $\rho^{(1,S)}$ is non-zero only in V_S and $\rho^{(1,F)}$ is non-zero only in V_F. Corresponding to this decomposition of $\rho^{(1)}$ we write $\phi^{(1)}$ as

$$\phi^{(1)} = \phi^{(1,S)} + \phi^{(1,F)}, \tag{25}$$

with the functions $\phi^{(1,S)}$ and $\phi^{(1,F)}$ defined to be solutions of the boundary value problems

$$\nabla^2\phi^{(1,S)} = \begin{cases} 4\pi G\rho^{(1,S)} & \mathbf{x}\in V_S \\ 0 & \mathbf{x}\in\mathbb{R}^3\setminus V_S \end{cases}, \tag{26}$$

and

$$\nabla^2\phi^{(1,F)} = \begin{cases} 4\pi G\rho^{(1,F)} & \mathbf{x}\in V_F \\ 0 & \mathbf{x}\in\mathbb{R}^3\setminus V_F \end{cases}, \tag{27}$$

subject to the boundary conditions

$$[\phi^{(1,S)}]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma, \tag{28}$$

$$[\hat{\mathbf{n}}\cdot\nabla\phi^{(1,S)} + 4\pi G\rho^{(0)}h^{(S)}]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma^{(S)}, \tag{29}$$

$$[\hat{\mathbf{n}}\cdot\nabla\phi^{(1,S)}]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma^{(F)}, \tag{30}$$

and

$$[\phi^{(1,F)}]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma, \tag{31}$$

$$[\hat{\mathbf{n}}\cdot\nabla\phi^{(1,F)}]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma^{(S)}, \tag{32}$$

$$[\hat{\mathbf{n}}\cdot\nabla\phi^{(1,F)} + 4\pi G\rho^{(0)}h^{(F)}]_{-}^{+} = 0, \quad \mathbf{x}\in\Sigma^{(F)}, \tag{33}$$

where we have defined $\Sigma^{(S)} = \Sigma_{SS}\cup\Sigma_{SF}\cup\Sigma_{FS}$, and have written $\Sigma^{(F)}$ for Σ_{FF}. The superscripts (S) and (F) on the boundary perturbations h have also been introduced to indicate whether a given boundary perturbation acts on a boundary in $\Sigma^{(S)}$ or $\Sigma^{(F)}$. Making use of these notations we can write eq. (15) as

$$-\nabla\hat{p}^{(1)} = \rho^{(0)}\nabla[\phi^{(1,S)} + \phi^{(1,F)}] + g\rho^{(1,F)}\hat{\mathbf{r}}, \tag{34}$$

where we have used the fact that $\nabla\phi^{(0)} = g\hat{\mathbf{r}}$ with $\hat{\mathbf{r}}$ the unit vector in the radial direction. Taking the cross product of this equation with $\hat{\mathbf{r}}$ we obtain the relation

$$\begin{aligned} 0 &= \hat{\mathbf{r}}\times\{\nabla\hat{p}^{(1)} + \rho^{(0)}\nabla[\phi^{(1,S)} + \phi^{(1,F)}] + g\rho^{(1,F)}\hat{\mathbf{r}}\} \\ &= \hat{\mathbf{r}}\times\{\nabla[\hat{p}^{(1)} + \rho^{(0)}(\phi^{(1,S)} + \phi^{(1,F)})] \\ &\quad - \partial_r\rho^{(0)}[\phi^{(1,S)} + \phi^{(1,F)}]\hat{\mathbf{r}}\} \\ &= \hat{\mathbf{r}}\times\nabla\{\hat{p}^{(1)} + \rho^{(0)}[\phi^{(1,S)} + \phi^{(1,F)}]\}, \end{aligned} \tag{35}$$

from which we readily deduce that the quantity $\hat{p}^{(1)} + \rho^{(0)}(\phi^{(1,S)} + \phi^{(1,F)})$ depends only upon the coordinate r. However, we know that $\hat{p}^{(1)}$, $\phi^{(1,S)}$ and $\phi^{(1,F)}$ all have zero mean over any spherical surface, so we conclude that the identity

$$\hat{p}^{(1)} = -\rho^{(0)}[\phi^{(1,S)} + \phi^{(1,F)}], \tag{36}$$

holds in V_F. The above equation shows that the aspherical part of the pressure perturbation in fluid-regions of the model is fully specified by knowledge of the perturbation in gravitational potential. From this relation and the boundary conditions for $\hat{p}^{(1)}$ on $\Sigma^{(F)}$ we also obtain the identity

$$h^{(F)} = -g^{-1}[\phi^{(1,S)} + \phi^{(1,F)}], \tag{37}$$

showing that the fluid–fluid boundary perturbations are also fully determined by the gravitational potential perturbation. Returning to eq. (34) we take the curl of both sides to obtain the relation

$$\begin{aligned} 0 &= \nabla \times \{\rho^{(0)}\nabla[\phi^{(1,S)} + \phi^{(1,F)}] + g\rho^{(1,F)}\hat{\mathbf{r}}\} \\ &= \partial_r \rho^{(0)}\mathbf{r} \times \nabla[\phi^{(1,S)} + \phi^{(1,F)}] + g\nabla\rho^{(1,F)} \times \hat{\mathbf{r}} \\ &= \hat{\mathbf{r}} \times \nabla\{\partial_r \rho^{(0)}[\phi^{(1,S)} + \phi^{(1,F)}] - g\rho^{(1,F)}\}. \end{aligned} \tag{38}$$

By an argument similar to that leading to eq. (36) we see that this equation implies the equality

$$\rho^{(1,F)} = g^{-1}\partial_r \rho^{(0)}[\phi^{(1,S)} + \phi^{(1,F)}], \tag{39}$$

showing that the density perturbation in fluid regions is also fully determined by the gravitational potential perturbation. Making use of eqs (39) and (37) the equation for $\phi^{(1,F)}$ can now be transformed into the equation

$$\nabla^2\phi^{(1,F)} = \begin{cases} 4\pi G g^{-1}\partial_r \rho^{(0)}[\phi^{(1,S)} + \phi^{(1,F)}] & \mathbf{x} \in V_F \\ 0 & \mathbf{x} \in \mathbb{R}^3 \setminus V_F \end{cases}, \tag{40}$$

subject to the boundary conditions

$$[\phi^{(1,F)}]_-^+ = 0, \quad \mathbf{x} \in \Sigma, \tag{41}$$

$$[\hat{\mathbf{n}} \cdot \nabla\phi^{(1,F)}]_-^+ = 0, \quad \mathbf{x} \in \Sigma^{(S)}, \tag{42}$$

$$[\hat{\mathbf{n}} \cdot \nabla\phi^{(1,F)} - 4\pi G g^{-1}\rho^{(0)}(\phi^{(1,S)} + \phi^{(1,F)})]_-^+ = 0, \quad \mathbf{x} \in \Sigma^{(F)}. \tag{43}$$

From the above results we conclude that in perturbing a spherically symmetric reference model with a hydrostatic equilibrium stress field the only free-parameters are as follows.

(i) The density perturbation in solid regions.
(ii) Boundary perturbations to solid–solid and fluid–solid boundaries.
(iii) Constant pressure perturbations in each connected component of the fluid regions.

Having specified these perturbations the density perturbation in fluid regions, the pressure perturbation in fluid regions, and the fluid–fluid boundary perturbations are fully determined.

An interesting consequence of the above analysis is that lateral variations in density or boundary topography in the mantle or inner core of the Earth will induce lateral variations in density within the fluid outer core. As discussed in detail by Wahr & de Vries (1989), such lateral variations in the outer core are consistent with the conclusions of Stevenson (1987) who employed fluid-dynamic arguments to show that within the outer core surfaces of constant material properties should closely coincide with surfaces of constant gravitational potential. Based upon Stevenson's arguments, the existence of lateral heterogeneities within the outer core not induced by lateral density or boundary variations in the mantle or inner core seems unlikely. Indeed, any such density variations would be inconsistent with our assumption of stresses being hydrostatic in the outer core.

3 CONSTRUCTING PARTICULAR SOLUTIONS OF THE EQUILIBRIUM EQUATIONS

In this section, we will be concerned with describing two ways of determining physically plausible particular solutions to the equilibrium equations. These equilibrium stress fields, once determined, can then be used as the stress field $\mathbf{T}_m$ occurring in eq. (1) of the introduction. In addressing this problem it will be useful to generalize and simplify the form of the equilibrium equations given in the previous section. To do this we consider the problem of finding a symmetric tensor field $\mathbf{T}$ defined in a compact region $V \subseteq \mathbb{R}^3$ satisfying the equation

$$\mathrm{Div}\mathbf{T} = \mathbf{f}, \tag{44}$$

for a given vector field $\mathbf{f}$, subject to the inhomogeneous boundary conditions

$$[\hat{\mathbf{n}} \cdot \mathbf{T}]_-^+ = \mathbf{t}, \tag{45}$$

with $\mathbf{t}$ a given vector field on Σ. Here V is now an arbitrary compact subset of $\mathbb{R}^3$ with smooth internal and external boundaries whose union is denoted by Σ.

3.1 Some notations and preliminary results

With the region V as above, we write $\mathscr{V}$ for the Hilbert space of real-valued vector fields defined in V that are square-integrable with respect to the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathscr{V}} = \int_V u_i v_i \,\mathrm{d}V. \tag{46}$$

Similarly, we write $\mathscr{T}$ for the Hilbert space of real-valued second order symmetric tensor fields defined in V that are square integrable with respect to the inner product

$$\langle \mathbf{S}, \mathbf{T} \rangle_{\mathscr{T}} = \int_V \frac{1}{2\mu} S_{ij} T_{ij} \,\mathrm{d}V, \tag{47}$$

where μ is a real-valued, everywhere positive (i.e. there exists a positive constant M such that $\mu(\mathbf{x}) > M$ for all $\mathbf{x} \in V$), and piecewise continuously differentiable function. The reason for introducing this weighting factor will be clarified below.

The divergence operator Div was introduced informally in Section 2. To define this operator more precisely we must specify the *domain* Dom(Div) of second order symmetric tensor fields on which it can act. To do this we let Dom(Div) be the linear subspace of $\mathscr{T}$ comprising those elements $\mathbf{T} \in \mathscr{T}$ for which Div $\mathbf{T}$ (defined in the sense of distributions) is an element of $\mathscr{V}$. Clearly, all elements $\mathbf{T} \in \mathscr{T}$ whose components have continuous first-order partial derivatives will be contained in Dom(Div). More generally, it may be shown using techniques from the theory of Sobolev spaces that Dom(Div) is dense in $\mathscr{T}$, and that the boundary tractions $\hat{\mathbf{n}} \cdot \mathbf{T}$ on Σ of elements $\mathbf{T} \in$ Dom(Div) are well-defined (e.g. Ting 1977; Georgescu 1980). Corresponding to the divergence operator we define its *image* to be the linear subspace of $\mathscr{V}$ given by

$$\mathrm{im(Div)} = \{\mathrm{Div}\mathbf{T} \in \mathscr{V} \mid \mathbf{T} \in \mathrm{Dom(Div)}\}, \tag{48}$$

and the *kernel* of Div to be the linear subspace of Dom(Div) defined by

$$\ker(\mathrm{Div}) = \{\mathbf{T} \in \mathrm{Dom(Div)} \mid \mathrm{Div}\mathbf{T} = 0\}. \tag{49}$$

Similar definitions of the image and kernel apply to any linear operator between two vector spaces.

The final concept we need to introduce is the adjoint of the divergence operator in the case that we restrict the action of Div to those elements of Dom(Div) satisfying the boundary condition $[\hat{\mathbf{n}} \cdot \mathbf{T}]_-^+ = \mathbf{0}$ on Σ. We denote the restriction of the divergence operator to this subspace as Div_0 and write Dom(Div_0) for its domain which may be shown to be dense in $\mathscr{T}$ (see the remarks following theorem 4.9 in Georgescu 1980). The adjoint operator Div_0^* corresponding to Div_0 is defined as follows (e.g. Yosida 1980, chapter VII, section 2, theorem 1): we say that a $\mathbf{u} \in \mathscr{V}$ is in $\mathrm{Dom}(\mathrm{Div}_0^*)$ if there exists a $\mathbf{u}^* \in \mathscr{T}$ such that for all $\mathbf{T} \in \mathrm{Dom}(\mathrm{Div}_0)$ we have

$$\langle \mathrm{Div}_0 \mathbf{T}, \mathbf{u}\rangle_{\mathscr{V}} = \langle \mathbf{T}, \mathbf{u}^*\rangle_{\mathscr{T}} . \tag{50}$$

For such $\mathbf{u} \in \mathrm{Dom}(\mathrm{Div}_0^*)$ we define $\mathrm{Div}_0^*\,\mathbf{u}$ to equal $\mathbf{u}^*$, and so can write

$$\langle \mathrm{Div}_0 \mathbf{T}, \mathbf{u}\rangle_{\mathscr{V}} = \langle \mathbf{T}, \mathrm{Div}_0^* \mathbf{u}\rangle_{\mathscr{T}}, \tag{51}$$

for all $\mathbf{T} \in \mathrm{Dom}(\mathrm{Div}_0)$. From this identity it may be shown using the divergence theorem that $\mathrm{Dom}(\mathrm{Div}_0^*)$ comprises the dense linear subspace of vector fields in $\mathscr{V}$ whose first-order partial derivatives (taken in the distributional sense) are square integrable, and that the action of Div_0^* on an element $\mathbf{u} \in \mathrm{Dom}(\mathrm{Div}_0^*)$ can be written

$$\mathrm{Div}_0^* \mathbf{u} = -2\mu \nabla_s \mathbf{u}, \tag{52}$$

where ∇_s denotes the 'symmetric gradient operator' which is given in component form by

$$(\nabla_s \mathbf{u})_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{53}$$

3.2 Orthogonal decomposition of the equilibrium stress fields

In this subsection, we describe an orthogonal decomposition for second order symmetric tensor fields which will allow us to determine a particular solution of the equilibrium equations. This decomposition theorem is a special case of a more general theorem for linear partial differential operators which may also be used to prove the Helmholtz decomposition theorem for vector fields and the Hodge decomposition theorem for differential forms (Berger & Ebin 1969; Cantor 1981; Voisin 2002). Particularly useful for our purposes are the results of Ting (1977) and Georgescu (1980) who consider the decomposition of the Hilbert space $\mathscr{T}$ in a compact region possessing a smooth boundary. The main result of interest is that the space $\mathscr{T}$ can be written as the orthogonal direct sum

$$\mathscr{T} = \ker(\mathrm{Div}_0) \oplus \mathrm{im}(\mathrm{Div}_0^*), \tag{54}$$

where the operators Div_0 and Div_0^* have been defined above (see theorem 3.1 of Ting 1997, and theorems 4.2 and 4.9 of Georgescu 1980). This result says that any element $\mathbf{T} \in \mathscr{T}$ can be written uniquely in the form

$$\mathbf{T} = -\mathrm{Div}_0^* \mathbf{u} + \mathbf{S}, \tag{55}$$

for some $\mathbf{u} \in \mathrm{Dom}(\mathrm{Div}_0^*)$ and $\mathbf{S} \in \ker(\mathrm{Div}_0)$, and that the two terms in the sum are orthogonal with respect to the inner product on $\mathscr{T}$ (we include a minus sign in the first term of this expression for later convenience). In particular, theorem 4.9 of Georgescu (1980) shows that if $\mathbf{T} \in \mathrm{Dom}(\mathrm{Div})$ then the vector field $\mathbf{u} \in \mathrm{Dom}(\mathrm{Div}_0^*)$ occurring in the above decomposition is such that $\mathrm{Div}_0^*\,\mathbf{u} \in \mathrm{Dom}(\mathrm{Div})$ (this result essentially means that the components of $\mathbf{u}$ have square-integrable partial derivatives upto second-order). Because of this we can substitute the above expression for $\mathbf{T} \in \mathrm{Dom}(\mathrm{Div})$ into the boundary value problem at the start of this section and, making use of the expression for Div_0^* given above, obtain the equation

$$\mathrm{Div}(2\mu \nabla_s \mathbf{u}) = \mathbf{f}, \tag{56}$$

with the boundary conditions

$$[2\mu \hat{\mathbf{n}} \cdot \nabla_s \mathbf{u}]_-^+ = \mathbf{t}, \tag{57}$$

on Σ. This boundary value problem for $\mathbf{u}$ has exactly the same form as the static linear elastic displacement problem in V for a material with shear moduls μ and bulk modulus $\kappa = \frac{2}{3}\mu$ subject to the body force $\mathbf{f}$ and surface tractions $\mathbf{t}$. We note that the minus sign in the first term in eq. (55) and the form of the weighting factor used in the inner product on $\mathscr{T}$ were chosen so that this direct correspondence would arise. Because μ has been assumed to be everywhere positive we can use standard existence and uniqueness theorems (e.g. Marsden & Hughes 1983, chapter 6, theorem 1.11) to show that a solution $\mathbf{u}$ to this boundary value problem exists so long as the identity

$$\int_V \mathbf{f} \cdot (\mathbf{a} + \mathbf{B}\mathbf{x})\,\mathrm{d}V = -\int_\Sigma \mathbf{t} \cdot (\mathbf{a} + \mathbf{B}\mathbf{x})\,\mathrm{d}S, \tag{58}$$

holds for all constant vector fields $\mathbf{a}$ and all constant anti-symmetric matrix fields $\mathbf{B}$. Physically this condition implies that the body forces and surface tractions apply no net force nor net torque to body. It may be shown that the solution $\mathbf{u}$ to this boundary value problem is defined uniquely up to the addition of an element of $\ker(\nabla_s)$ (i.e. a rigid rotation or translation). Consequently, solution of this boundary value problem yields a unique equilibrium stress field $-2\mu\nabla_s \mathbf{u}$.

3.3 Minimum equilibrium stress fields

If we write $||\mathbf{T}||_{\mathscr{T}}$ for the norm on $\mathscr{T}$ induced by the given inner product, then, from eq. (54) we see that the squared norm of an equilibrium stress field $\mathbf{T}$ is given by

$$||\mathbf{T}||^2_{\mathscr{T}} = ||-\mathrm{Div}_0^* \mathbf{u}||^2_{\mathscr{T}} + ||\mathbf{S}||^2_{\mathscr{T}}, \tag{59}$$

where $\mathbf{S} = \mathbf{T} + \mathrm{Div}_0^*\,\mathbf{u} \in \ker(\mathrm{Div}_0)$, and $\mathbf{u}$ is a solution of the boundary value problem just described. From this relation it is clear that the equilibrium stress field $-\,\mathrm{Div}_0^*\,\mathbf{u}$ is the solution of the equilibrium equations that minimizes the norm-functional. This result may also be established in a less formal manner by considering the variational problem to minimize the functional

$$I = \frac{1}{2}||\mathbf{T}||^2_{\mathscr{T}}, \tag{60}$$

subject to the constraint that $\mathbf{T} \in \mathrm{Dom}(\mathrm{Div})$ is also a solution of the equilibrium equations. To solve this problem we introduce a Lagrange multiplier vector field $\mathbf{u} \in \mathrm{Dom}(\mathrm{Div}_0^*)$, and construct the augmented functional

$$I' = \frac{1}{2}||\mathbf{T}||^2_{\mathscr{T}} + \langle \mathbf{u}, \mathrm{Div}\mathbf{T} - \mathbf{f}\rangle_{\mathscr{V}} , \tag{61}$$

which incorporates the constraint that $\mathbf{T}$ be a solution of the equilibrium equations. The first variation of functional with respect to the admissible variations $\delta\mathbf{T} \in \mathrm{Dom}(\mathrm{Div}_0)$ and $\delta\mathbf{u} \in \mathrm{Dom}(\mathrm{Div}_0^*)$ is found to be

$$\begin{aligned} \delta I' &= \langle \mathbf{T}, \delta\mathbf{T}\rangle_{\mathscr{T}} + \langle \mathbf{u}, \mathrm{Div}_0 \delta\mathbf{T}\rangle_{\mathscr{V}} + \langle \delta\mathbf{u}, \mathrm{Div}\mathbf{T} - \mathbf{f}\rangle_{\mathscr{V}} \\ &= \langle \mathbf{T} + \mathrm{Div}_0^* \mathbf{u}, \delta\mathbf{T}\rangle_{\mathscr{T}} + \langle \delta\mathbf{u}, \mathrm{Div}\mathbf{T} - \mathbf{f}\rangle_{\mathscr{V}} , \end{aligned} \tag{62}$$

from which we see, using the density of $\mathrm{Dom}(\mathrm{Div}_0)$ in $\mathscr{T}$ and of $\mathrm{Dom}(\mathrm{Div}_0^*)$ in $\mathscr{V}$, that the vanishing of $\delta I'$ for all $\delta\mathbf{T} \in \mathrm{Dom}(\mathrm{Div}_0)$ and $\delta\mathbf{u} \in \mathrm{Dom}(\mathrm{Div}_0^*)$ does, as expected, lead to the relation $\mathbf{T} = -\mathrm{Div}_0^*\,\mathbf{u}$, along with the boundary value problem for $\mathbf{u}$ described above. We call the resulting equilibrium stress field the 'minimum equilibrium stress field' for the earth model. However, because this equilibrium stress field depends upon the choice of inner product on $\mathscr{T}$ (as expressed by the function μ) it is important to remember that this stress field is only a 'minimum' with respect to the given norm on $\mathscr{T}$, and not in any absolute sense. It is not difficult to see that the equilibrium stress field obtained by the above method does not change if μ is replaced by some positive scalar multiple of itself, though it does depend upon any spatial variations of μ. This property allows us to build in further *a priori* preferences into the 'minimum equilibrium stress field', and was our primary motivation for introducing the term μ into the problem. For example, we might expect that the departure of the equilibrium stress field from its hydrostatic reference value would be larger (relative to the magnitude of the ambient hydrostatic stress) in the lithosphere than in the underlying mantle due to the occurrence of tectonic deformation. In this case, we could incorporate this preference into the minimization problem by setting the magnitude of μ in the lithosphere larger than in the mantle, and so down-weighting the effects of lithospheric stresses in the minimization problem.

A modification of the above approach is to determine a solution of the equilibrium equations which minimizes the norm of its deviatoric component. Such an equilibrium stress field is of physical interest because it is likely that the deviatoric part of the equilibrium stress field in the Earth is relatively small due to its magnitude being limited by the strength of rocks in the Earth's interior. We write $\pi : \mathscr{T} \to \mathscr{T}$ for the bounded linear operator mapping symmetric second order tensor fields onto the linear subspace of trace-free symmetric second order tensor fields given by

$$\pi \mathbf{T} = \mathbf{T} - \frac{1}{3}\mathrm{tr}(\mathbf{T})\mathbf{1}, \tag{63}$$

where $\mathrm{tr}(\mathbf{T}) = T_{ii}$ is the trace of a tensor field. It is clear that π is 'idempotent' (i.e. $\pi\pi = \pi$) so that it defines a projection operator. Moreover, a simple calculation shows that

$$\langle \pi \mathbf{S}, \mathbf{T} \rangle_{\mathscr{T}} = \langle \mathbf{S}, \pi \mathbf{T} \rangle_{\mathscr{T}}\,, \tag{64}$$

for any $\mathbf{S}, \mathbf{T} \in \mathscr{T}$, so that π is 'self-adjoint' with respect to the inner product on $\mathscr{T}$. We wish to find the solution of the equilibrium equations that minimizes the functional

$$J = \frac{1}{2}||\pi \mathbf{T}||^2_{\mathscr{T}}. \tag{65}$$

To solve this problem we again introduce a Lagrange multiplier vector field $\mathbf{u} \in \mathrm{Dom}(\mathrm{Div}_0^*)$, and form the augmented functional

$$J' = \frac{1}{2}||\pi \mathbf{T}||^2_{\mathscr{T}} + \langle \mathbf{u}, \mathrm{Div}\mathbf{T} - \mathbf{f} \rangle_{\mathscr{V}}\,, \tag{66}$$

to incorporate the constraint that $\mathbf{T}$ be an equilibrium stress field. The first variation of this functional with respect to the admissible variations $\delta\mathbf{T} \in \mathrm{Dom}(\mathrm{Div}_0)$ and $\delta\mathbf{u} \in \mathrm{Dom}(\mathrm{Div}_0^*)$ is found to be

$$\begin{aligned} \delta J' &= \langle \pi \mathbf{T}, \pi \delta \mathbf{T} \rangle_{\mathscr{T}} + \langle \mathbf{u}, \mathrm{Div}_0 \delta \mathbf{T} \rangle_{\mathscr{V}} + \langle \delta \mathbf{u}, \mathrm{Div}\mathbf{T} - \mathbf{f} \rangle_{\mathscr{V}} \\ &= \langle \pi \mathbf{T} + \mathrm{Div}_0^* \mathbf{u}, \delta \mathbf{T} \rangle_{\mathscr{T}} + \langle \delta \mathbf{u}, \mathrm{Div}\mathbf{T} - \mathbf{f} \rangle_{\mathscr{V}}\,, \end{aligned} \tag{67}$$

where we have made use of the fact that π is self-adjoint and idempotent. The vanishing of $\delta J'$ for arbitrary $\delta \mathbf{T} \in \mathrm{Dom}(\mathrm{Div}_0)$ implies the relation

$$\pi \mathbf{T} = -\mathrm{Div}_0^* \mathbf{u}, \tag{68}$$

which in turn implies that

$$\mathrm{tr}(\mathrm{Div}_0^* \mathbf{u}) = 0. \tag{69}$$

Making use of the expression for the adjoint operator Div_0^* given above, we see that the first of these equations can be written

$$\pi \mathbf{T} = 2\mu \nabla_s \mathbf{u}, \tag{70}$$

while the second becomes

$$\mathrm{div}\mathbf{u} = 0. \tag{71}$$

Because eq. (70) only specifies the trace-free part of the stress tensor $\mathbf{T}$, it follows that the full relation between $\mathbf{T}$ and $\mathbf{u}$ must take the form

$$\mathbf{T} = -p\mathbf{1} + 2\mu \nabla_s \mathbf{u}, \tag{72}$$

where p is a scalar field which is defined such that $p = -\frac{1}{3}\mathrm{tr}(\mathbf{T})$ but is otherwise unconstrained. Inspection of eqs (71) and (72) shows that the relationship between the stress tensor $\mathbf{T}$ and the Largrange multiplier field $\mathbf{u}$ has exactly the same form as the constitutive equation of an incompressible Newtonian fluid with velocity vector $\mathbf{u}$, pressure p, and viscosity μ (e.g. Batchelor 1967). Assuming that the fields $\mathbf{u}$ and p are sufficiently well behaved, we can insert these expressions into the boundary value problem at the start of this section to obtain the equation

$$-\nabla p + \mathrm{Div}(2\mu \nabla_s \mathbf{u}) = \mathbf{f}, \tag{73}$$

which is subject to the incompressibility condition in eq. (71) and to the boundary conditions

$$[-p\hat{\mathbf{n}} + 2\mu \hat{\mathbf{n}} \cdot \nabla_s \mathbf{u}]_-^+ = \mathbf{t}, \tag{74}$$

on Σ. This boundary value problem is identical to the steady-state Navier–Stokes equations for an incompressible viscous fluid subject to the given body forces and surface tractions. Using existence and uniqueness theorems for the steady-state Navier–Stokes equations (e.g. Sohr 2000) we can conclude that solution of this problem leads to a uniquely determined equilibrium stress field so long as the conditions on $\mathbf{f}$ and $\mathbf{t}$ given in eq. (58) hold. We refer to the resulting equilibrium stress field as the 'minimum deviatoric equilibrium stress field' for the earth model. As with the case of the 'minimum equilibrium stress field', the equilibrium stress field obtained by solving this problem does not depend on the absolute magnitude of μ, but only upon its spatial variations. Because of this, we can again make use of spatial variations in μ to incorporate further *a priori* preferences about the form of the equilibrium stress field into the minimization problem.

We conclude this section by describing a particularly interesting property of the minimum deviatoric equilibrium stress field. To do this we first recall some basic facts about the elastic tensor of an earth model with a non-zero equilibrium stress field following the discussion given in section 3.6 of Dahlen & Tromp (1998). Let us denote by $\mathbf{s}$ the infinitesimal displacement vector describing the deformation of the earth model away from its equilibrium configuration, and by $\mathbf{T}^{L1}$ the incremental Lagrangian stress tensor resulting from this deformation (though there are a number of other measures of stress that can be used, the incremental Lagrangian stress tensor is most useful for our present purposes). It may be shown that $\mathbf{s}$ and $\mathbf{T}^{L1}$ are related by the constitutive equation

$$T_{ij}^{L1} = \Upsilon_{ijkl} s_{k,l}, \tag{75}$$

where Υ_{ijkl} is a tensor with the symmetries $\Upsilon_{ijkl} = \Upsilon_{klij}$ (see eqs 3.121 and 3.125 of Dahlen & Tromp 1998). It may be shown that Υ_{ijkl} can be written in the form

$$\begin{aligned} \Upsilon_{ijkl} &= \Gamma_{ijkl} \\ &\quad + \frac{1}{2}(T_{kl}\delta_{ij} - T_{ij}\delta_{kl} + T_{ik}\delta_{jl} - T_{jl}\delta_{ik} + T_{jk}\delta_{il} - T_{il}\delta_{jk}), \end{aligned} \tag{76}$$

where Γ_{ijkl} is a tensor possessing the symmetries

$$\Gamma_{ijkl} = \Gamma_{jikl} = \Gamma_{ijlk} = \Gamma_{klij}, \tag{77}$$

and where T_{ij} denotes the components of the equilibrium stress field in the earth model (see eq. 3.141 of Dahlen & Tromp 1998, and also eq. 4 of Dahlen 1972c). This expression can be regarded as splitting the elastic tensor Υ_{ijkl} into the sum of an 'intrinsic elastic tensor' Γ_{ijkl} and a second term which depends explicitly upon the equilibrium stress field. It is not difficult to show that the second term on the right-hand side of eq. (76) does not depend on the hydrostatic component of the equilibrium stress field, so that we can replace T_{ij} in the above expression with its deviatoric component for which we write τ_{ij}. We recall that the elastic tensor Υ_{ijkl} is said to be 'isotropic' if it takes the form

$$\Upsilon_{ijlk} = \left(\kappa - \frac{2}{3}\mu\right)\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{78}$$

where κ and μ are, respectively, the bulk and shear moduli of the earth model. When this condition is not met we say that Υ_{ijkl} is 'anisotropic'. From eq. (76) we see that anisotropy in Υ_{ijkl} can arise from either 'intrinsic anisotropy' of the tensor Γ_{ijkl} arising from, for example, the preferential alignment of crystallographic axes, or from 'stress-induced anisotropy' due to the deviatoric component of the equilibrium stress field. A simple measure of the magnitude of any stress-induced anisotropy in the earth mode is obtained by integrating $(\Upsilon_{ijkl} - \Gamma_{ijkl})\,(\Upsilon_{ijkl} - \Gamma_{ijkl})$ over the earth model, and a routine calculation shows that

$$(\Upsilon_{ijkl} - \Gamma_{ijkl})(\Upsilon_{ijkl} - \Gamma_{ijkl}) = \frac{9}{4}\tau_{ij}\tau_{ij}. \tag{79}$$

This implies that the equilibrium stress field which minimizes the magnitude of stress-induced anisotropy in the earth model (in the sense defined above) is precisely the 'minimum deviatoric equilibrium stress field' in the case that the weighting function μ in the definition of the scalar product on $\mathscr{T}$ is equal to an arbitrary positive constant.

4 CONSTRUCTING DIVERGENCE-FREE TENSOR FIELDS IN SPHERICALLY SYMMETRIC EARTH MODELS

In this section, we turn to considering how elements of the vector space $\ker(\mathrm{Div}_0)$ can be practically constructed. The results of Section 2 show that for our purposes such tensor fields need only be generated in the solid regions of a spherically symmetric reference earth model. Taking PREM of Dziewonski & Anderson (1981) as an example, we suppose that our earth model has a solid inner core, a fluid outer core, a solid mantle and crust, and a fluid ocean. Consequently we require a method for generating elements of $\ker(\mathrm{Div}_0)$ within the inner core and solid mantle of the earth model. For simplicity we focus attention on the construction of elements of $\ker(\mathrm{Div}_0)$ within the mantle and crust of such an earth model. Essentially the same method can be used in the inner core, though in this case we must deal with the added complication of insuring that the tensor fields are regular at the center of the earth model.

We suppose that the mantle and crust of the earth model occupies the spherical shell with inner radius r_1 and outer radius r_n, and that there are $n-2$ internal spherical boundaries with radii $r_1 < r_2 < \ldots < r_{n-1} < r_n$. Denoting this spherical shell by V and the union of all internal and external boundaries by Σ, we then wish to generate symmetric second-order tensor fields $\mathbf{T} \in \mathrm{Dom}(\mathrm{Div}_0)$. To do this we use the method of Backus (1967) specialized to the case of divergence- and traction-free tensor fields. In doing this we do not, however, make use of Backus's representation theorem for the tensor fields in terms of six scalar potential functions (Backus 1966). Instead, we express our results in terms of the generalized spherical harmonic functions of Phinney & Burridge (1973) whose use is now more common (see also Dahlen & Tromp 1998, Appendix C). In this formalism an arbitrary element $\mathbf{T} \in \mathrm{Dom}(\mathrm{Div}_0)$ can be expanded in terms of generalized spherical harmonics as

$$\mathbf{T}(r,\theta,\varphi) = \sum_{lm} T^{\alpha\beta}_{lm}(r)Y^N_{lm}(\theta,\varphi)\hat{\mathbf{e}}_\alpha\hat{\mathbf{e}}_\beta, \tag{80}$$

where the coefficient functions $T^{\alpha\beta}_{lm}$ are piecewise continuously differentiable functions of r, summation over repeated Greek indices is implied in the range $\{-,0,+\}$, $\mathbf{e}_\alpha$ is a unit vector in the canonical basis, and Y^N_{lm} is a generalized spherical harmonic of degree l, order m, and upper index $N = \alpha + \beta$. Because $\mathbf{T}$ is a symmetric tensor field the radial expansion coefficients $T^{\alpha\beta}_{lm}$ have the symmetry

$$T^{\alpha\beta}_{lm} = T^{\beta\alpha}_{lm}. \tag{81}$$

In addition, the requirement that $\mathbf{T}$ be real-valued implies

$$T^{-\alpha-\beta}_{l-m} = (-1)^m\,\bar{T}^{\alpha\beta}_{lm}, \tag{82}$$

where the overbar denotes complex-conjugation (this relationship follows from (C.107) of Dahlen & Tromp 1998). Due to these conditions we see that for each $l \geq 0$ we need only consider values of m in the range $0, 1, \ldots, l$, and that for each l and m there are only six independent coefficient functions $T^{\alpha\beta}_{lm}$.

The condition that $\mathrm{Div}\,\mathbf{T} = \mathbf{0}$ is expressed for each value of l and m in terms of the three ordinary differential equations

$$\partial_r T^{-0}_{lm} + 3r^{-1}T^{-0}_{lm} - \Omega^0_l r^{-1}T^{-+}_{lm} - \Omega^2_l r^{-1}T^{--}_{lm} = 0, \tag{83}$$

$$\partial_r T^{00}_{lm} + 2r^{-1}\left(T^{00}_{lm} + T^{-+}_{lm}\right) - \Omega^0_l r^{-1}\left(T^{-0}_{lm} + T^{0+}_{lm}\right) = 0, \tag{84}$$

$$\partial_r T^{0+}_{lm} + 3r^{-1}T^{0+}_{lm} - \Omega^0_l r^{-1}T^{-+}_{lm} - \Omega^2_l r^{-1}T^{++}_{lm} = 0, \tag{85}$$

where $\Omega^N_l = \sqrt{(l+N)(l-N+1)/2}$. Similarly, the traction free-boundary conditions for the problem become

$$\left[T^{-0}_{lm}(r_i)\right]^+_- = 0, \tag{86}$$

$$\left[T^{00}_{lm}(r_i)\right]^+_- = 0, \tag{87}$$

$$\left[T^{0+}_{lm}(r_i)\right]^+_- = 0, \tag{88}$$

where $i = 1, \ldots, n$, and it is understood that the radial coefficient functions vanish for r outside the interval $[r_1, r_n]$.

At this stage it will be useful at this stage to introduce six new radial coefficient functions for each l and m through the equations

$$T^{--}_{lm} = 2\Omega^0_l\Omega^2_l\left(M_{lm} - \mathrm{i}N_{lm}\right), \tag{89}$$

$$T^{-0}_{lm} = \Omega^0_l\left(S_{lm} - \mathrm{i}T_{lm}\right), \tag{90}$$

$$T^{-+}_{lm} = L_{lm}, \tag{91}$$

$$T^{00}_{lm} = P_{lm}, \tag{92}$$

$$T^{0+}_{lm} = \Omega^0_l\left(S_{lm} + \mathrm{i}T_{lm}\right), \tag{93}$$

$$T^{++}_{lm} = 2\Omega^0_l\Omega^2_l\left(M_{lm} + \mathrm{i}N_{lm}\right), \tag{94}$$

which correspond to the toroidal and poloidal combinations of Phinney & Burridge (1973). In terms of these new functions, eqs (83)–(85) are seen to decouple into the 2-D system

$$\partial_r P_{lm} + 2r^{-1}\left(P_{lm} + L_{lm}\right) - \zeta^2 r^{-1}S_{lm} = 0, \tag{95}$$

$$\partial_r S_{lm} + 3r^{-1} S_{lm} - r^{-1} L_{lm} - (\zeta^2 - 2) r^{-1} M_{lm} = 0, \tag{96}$$

and the 1-D system

$$\partial_r T_{lm} + 3r^{-1} T_{lm} - (\zeta^2 - 2) r^{-1} N_{lm} = 0, \tag{97}$$

where $\zeta = \sqrt{l(l+1)}$, while the boundary conditions become

$$[P_{lm}(r_i)]_-^+ = 0, \tag{98}$$

$$[S_{lm}(r_i)]_-^+ = 0, \tag{99}$$

$$[T_{lm}(r_i)]_-^+ = 0, \tag{100}$$

for $i = 1, \ldots, n$. We call any tensor field for which $T_{lm} = N_{lm} = 0$ a 'spheroidal tensor field', and any tensor field for which $P_{lm} = S_{lm} = L_{lm} = M_{lm} = 0$ a 'toroidal tensor field'. In deriving the eqs (95)–(97) we have assumed that $l \geq 1$ so that $\Omega_l^0 \neq 0$. In the special case $l = 0$ only T_{00}^{00} and T_{00}^{-+} can be non-zero due to the fact that $Y_{lm}^N = 0$ if $|N| > l$. From this we see that in the case $l = 0$ eqs (83) and (85) are satisfied identically, while eq. (84) becomes

$$\partial_r P_{00} + 2r^{-1}(P_{00} + L_{00}) = 0. \tag{101}$$

Let us first consider how to generate toroidal elements of ker(Div$_0$). It is clear from eq. (97) that if we let T_{lm} be an arbitrarily piecewise differentiable function in $[r_1, r_n]$ we can obtain a divergence-free toroidal tensor field by setting

$$N_{lm} = \frac{1}{\zeta^2 - 2}\left(r\partial_r T_{lm} + 3T_{lm}\right), \tag{102}$$

where we have assumed that $l \geq 2$. Moreover, by requiring that the given function T_{lm} be continuous across all internal boundaries $r_2, \ldots, r_{n-1}$, and that it vanish at r_1 and r_n, we obtain a toroidal element of ker(Div$_0$). We have seen above that there are no toroidal tensor fields in the case $l = 0$. Similarly in the case $l = 1$ it follows from $Y_{1m}^{\pm 2} = 0$ that $N_{1m} = 0$, so we can solve eq. (97) to obtain $T_{1m} = cr^{-3}$ where c is an arbitrary constant. However, it is clear that $T_{1m} = cr^{-3}$ cannot satisfy the boundary conditions $T_{1m}(r_1) = T_{1m}(r_n) = 0$ for any non-zero value of the constant c, and we conclude that there are no non-trivial toroidal elements of ker(Div$_0$) for $l = 1$.

We can construct spheroidal elements of ker(Div$_0$) by a similar process. From eqs (95) and (86) it is readily seen that L_{lm} and M_{lm} can be expressed in terms of P_{lm} and S_{lm} by the expressions

$$L_{lm} = \frac{1}{2}\left[\zeta^2 S_{lm} - r\partial_r P_{lm} - 2P_{lm}\right], \tag{103}$$

$$M_{lm} = \frac{1}{2(\zeta^2 - 2)}\left[r\partial_r P_{lm} + 2P_{lm} + 2r\partial_r S_{lm} - (\zeta^2 - 6) S_{lm}\right], \tag{104}$$

where we have again assumed $l \geq 2$. From this we see that if we arbitrarily specify P_{lm} and S_{lm} to be piecewise differentiable functions in $[r_1, r_n]$ such that they are continuous at each internal boundary $r_2, \ldots, r_{n-1}$ and vanish at the end points r_1 and r_n, then we obtain a spheroidal element of ker(Div$_0$). In the case $l = 0$ we have seen that only eq. (101) need be satisfied, and this may be achieved by setting

$$L_{00} = -\frac{1}{2}\left(r\partial_r P_{00} + 2P_{00}\right), \tag{105}$$

for any function P_{00} that is continuous, piecewise differentiable in $[r_1, r_n]$, and vanishes at the endpoints of the interval. In the case $l = 1$ we must have $M_{1m} = 0$, and eqs (95) and (86) reduce to

$$\partial_r P_{1m} + 2r^{-1}(P_{1m} + L_{1m}) - 2r^{-1} S_{1m} = 0, \tag{106}$$

$$\partial_r S_{1m} + 3r^{-1} S_{1m} - r^{-1} L_{1m} = 0. \tag{107}$$

Solving the second of these equations for L_{1m} leads to

$$L_{1m} = r\partial_r S_{1m} + 3S_{1m}, \tag{108}$$

which, when substituted in eq. (95), gives

$$\partial_r (P_{1m} + 2S_{1m}) + 2r^{-1}(P_{1m} + 2S_{1m}) = 0. \tag{109}$$

Using the required continuity of P_{1m} and S_{1m}, we see that this latter equation implies that

$$P_{1m} + 2S_{1m} = cr^{-2}, \tag{110}$$

with c some constant. However, the boundary conditions on P_{1m} and S_{1m} are such that this constant c must equal zero, and we conclude that the identity

$$P_{1m} = -2S_{1m}, \tag{111}$$

must hold for all spheroidal elements of ker(Div) in the case $l = 1$. If we choose the function S_{1m} in eq. (111) to be continuous, piecewise differentiable in $[r_1, r_n]$, and to vanish at r_1 and r_n, then the above formulae give a spheroidal element of ker(Div$_0$) for $l = 1$.

Let us denote by $\mathscr{C}$ the vector space of complex-valued scalar functions defined on $[r_1, r_n]$ that are piecewise continuously differentiable, vanish at the endpoints r_1 and r_n, and are continuous across the internal boundaries $r_2, \ldots, r_{n-1}$. In terms of these functions we can summarize the above results as follows:

(i) For $l = 0$ there is one spheroidal element of ker(Div$_0$) corresponding to each element of $\mathscr{C}$.
(ii) For $l = 1$ and for each $m = 0, 1$ there is one spheroidal element of ker(Div$_0$) corresponding to each element of $\mathscr{C}$.
(iii) For $l \geq 2$ and each $m = 0, 1, \ldots, l$ there is one toroidal element of ker(Div$_0$) corresponding to each element of $\mathscr{C}$, and there is one spheroidal element of ker(Div$_0$) corresponding to each pair of elements of $\mathscr{C}$.

To apply these results practically we must be able to generate a number of elements of $\mathscr{C}$. Clearly, there are many possible ways of doing this. For example, elements of $\mathscr{C}$ can be readily produced using cubic spline interpolation, or with Lagrange polynomial interpolation.

5 DISCUSSION

In this paper, we have described a method of parametrizing the possible equilibrium stress fields in a slightly laterally heterogeneous earth model. The primary difference between our method and that described by Backus (1967) is that we have considered how to construct particular solution of the equilibrium equations possessing desirable physical characteristics. In particular, we have shown how to construct the equilibrium stress field possessing the smallest norm with respect to a given inner product, and the equilibrium stress field whose deviatoric component has smallest norm with respect to a given inner product. These particular solutions of the equilibrium equations have a number of potential applications. For example, we have seen that the minimum deviatoric equilibrium stress field is the solution of the equilibrium equations which leads to the smallest (in the sense previously defined) amount of stress-induced anisotropy in the earth model. Consequently, determination of the minimum deviatoric equilibrium stress field provides a useful 'lower-bound' on the amount of stress-induced anisotropy associated with a given laterally heterogeneous earth model.

We have not discussed in detail the practical implementation of some aspects of the the theory presented. In particular, we have not described a method for calculating the perturbations to the gravitational potential discussed in Section 2, nor a method for calculating either the minimum equilibrium stress field or the minimum deviatoric equilibrium stress field described in Section 3. However, the calculations required are fairly straightforward and can be performed using a range of existing numerical methods. For example, calculation of the perturbations to the gravitational potential essentially requires the solution of Poisson's equation in a sphere which is most easily done using spherical harmonic expansions. Similarly, calculation of the minimum equilibrium stress field or the minimum deviatoric equilibrium stress field requires, respectively, the solution of either a static linear elastic boundary value problem, or a steady-state incompressible viscous flow problem, and in each case there exists a number of possible methods of solution. In particular, if, as would seem sensible, we assume that the weighting factor μ occurring in the inner product on $\mathscr{T}$ depends only on the radial coordinate, then either of these problem can be efficiently solved by expanding the solution in spherical harmonics and numerically integrating the resulting system of ordinary differential equations (see, for example, Pollitz 1996; Forte 2007, for the solution of similar problems).

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