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Normal mode splitting due to inner core attenuation anisotropy

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SUMMARY

The Earth's inner core displays transverse velocity anisotropy with cylindrical symmetry, which causes the anomalous zonal splitting of inner core sensitive normal modes. In this paper, we extend existing theory for calculating normal mode splitting from models of cylindrical velocity anisotropy to include models of anelastic attenuation anisotropy. Furthermore, we derive the equations that can be used to rewrite the attenuation anisotropy parameters natural to normal mode considerations in terms of body wave attenuation anisotropy for compressional and shear waves.

Key words: Core, outer core and inner core; Elasticity and anelasticity; Surface waves and free oscillations; Seismic anisotropy; Seismic attenuation; Theoretical seismology.

1 INTRODUCTION

Anomalous zonal splitting of inner core sensitive normal modes was first observed by Masters & Gilbert (1981), and later interpreted in terms of inner core anisotropy, with the fast axis aligned with the Earth's rotation axis (e.g. Woodhouse *et al.* 1986). In splitting function observations, inner core anisotropy manifests itself as anomalously large zonal coefficients, observed only for modes with inner core sensitivity (Woodhouse *et al.* 1986; Giardini *et al.* 1988; Li *et al.* 1991; He & Tromp 1996; Romanowicz *et al.* 1996; Durek & Romanowicz 1999; Deuss *et al.* 2010, 2013). As the splitting function coefficients previously considered have all been elastic, the resulting anisotropy is also in the elastic part. Such analyses have given rise to several models of velocity anisotropy of the inner core (Woodhouse *et al.* 1986; Li *et al.* 1991; Tromp 1993; Romanowicz *et al.* 1996; Durek & Romanowicz 1999; Ishii *et al.* 2002; Beghein & Trampert 2003).

Recently, we have extended the normal mode splitting function technique to include not only elastic, but also anelastic, splitting function coefficients (Mäkinen & Deuss 2013). Akin to the elastic splitting function coefficients, these are found to be predominantly zonal for inner core sensitive modes. The theoretical formalism required to interpret the elastic zonal splitting function coefficients in terms of cylindrical velocity anisotropy was set up by Woodhouse *et al.* (1986) and Tromp (1995). In this paper, we extend their theoretical considerations to attenuation anisotropy. In doing so, we develop a theoretical framework for calculating zonal anelastic splitting function coefficients for anisotropy in anelastic attenuation. *P*-wave attenuation anisotropy in the inner core has been hinted at in some body wave studies (e.g. Souriau & Romanowicz 1996, 1997; Oreshin & Vinnik 2004; Yu & Wen 2006). In developing a theoretical formalism to use normal mode splitting function coefficients for elucidating inner core attenuation anisotropy, we enable comparisons between body wave and normal mode attenuation anisotropy observations, as well as the building of global inner core attenuation anisotropy models using zonal normal mode splitting function observations.

2 SPLITTING OF NORMAL MODES DUE TO ANISOTROPY

In this section, we consider the splitting of the Earth's normal modes due to anisotropy in both the elastic and the anelastic contributions. This is an extension of the purely elastic theory discussed by Tromp (1995). Anelasticity can readily be added by allowing imaginary parts to the (usually real) Love parameters A, C, L, N and F (Love 1927). We focus on spheroidal modes, denoted ${}_nS_l$, with l the angular order and n the overtone number (or radial order). Each normal mode, or multiplet, ${}_nS_l$ comprises $2l + 1$ singlets, labelled using the azimuthal order m, where $m = -l \ldots, 0, \ldots, l$. In a spherical, non-rotating, elastic and isotropic (SNREI) Earth model the $2l + 1$ singlets are degenerate, that is, they share the same frequency and quality factor. In a non-SNREI Earth, this denegeracy is lifted, and the $2l+1$ singlets split, resulting in different singlet frequencies and quality factors. The splitting of singlets can be described using the splitting matrix $\mathbf{M}$. Here we only consider self-coupling, that is, a given singlet is permitted to couple to singlets within the same multiplet ${}_nS_l$, and no coupling between singlets of different multiplets is allowed.

Following Woodhouse & Girnius (1982) and Deuss & Woodhouse (2001), the splitting contribution of a particular isolated multiplet to an observable seismogram $u(t)$ is

$$u(t) = \mathrm{Re}\left[\mathbf{r} \cdot e^{i\sqrt{\mathbf{M}}t} \cdot \mathbf{s}\right], \tag{1}$$

where t is the time; $\mathbf{s}$ is the source vector and depends on the source moment tensor; $\mathbf{r}$ is the receiver vector and depends on the orientation and response of the recording instrument; and $\mathbf{M}$ is the splitting matrix. $\mathbf{M}$ is a $(2l + 1) \times (2l + 1)$ complex matrix; it contains contributions due to rotation, the Earth's ellipticity, and heterogeneity in Earth structure (Dahlen 1968; Woodhouse & Dahlen 1978). The anisotropy contribution, $\mathbf{A}$, to the total splitting matrix $\mathbf{M}$ for two singlets of azimuthal orders m and m' is given by (Woodhouse & Dahlen 1978; Li *et al.* 1991)

$$A_{mm'} = \int_V \mathbf{E}_{m'} : \mathbf{\Lambda} : \mathbf{E}_m^* \mathrm{d}\mathbf{r}^3, \tag{2}$$

where the integration is over the volume V of the Earth, and $\mathbf{\Lambda}$ is a general viscoelastic tensor, similar to the elastic tensor used by Tromp (1995) but now comprising both real (superscript R, elastic) and imaginary (superscript I, anelastic) parts:

$$\mathbf{\Lambda} = \mathbf{\Lambda}^R + i\mathbf{\Lambda}^I. \tag{3}$$

The singlet strain tensor $\mathbf{E}_m$ above is found in terms of the displacement eigenfunctions $\mathbf{u}_m$ of the $2l + 1$ singlets comprising a given multiplet:

$$\mathbf{E}_m = \frac{1}{2}\left[\nabla\mathbf{u}_m + (\nabla\mathbf{u}_m)^T\right]. \tag{4}$$

To simplify the mathematics involved, we express the tensors $\mathbf{u}_m$, $\mathbf{E}_m$ and $\mathbf{\Lambda}$ above in the canonical basis defined by Phinney & Burridge (1973). This basis is summarized in Appendix A. We start by writing the eigenfunction $\mathbf{u}_m$ in terms of its components $u^\alpha(r)$ with respect to the canonical basis vectors $\hat{\mathbf{e}}_\alpha$ (with $\alpha = -1, 0, +1$), using the fully normalized generalized spherical harmonics $Y_{lm}^\alpha(\theta, \phi)$ (Edmonds 1960):

$$\mathbf{u}_m(\mathbf{r}) = \sum_\alpha u^\alpha(r) Y_{lm}^N(\theta, \phi)\hat{\mathbf{e}}_\alpha, \tag{5}$$

where $N = \alpha$. With respect to the same canonical basis vectors, the singlet strain tensor $\mathbf{E}_m$ becomes

$$\mathbf{E}_m(\mathbf{r}) = \sum_{\alpha,\beta} E^{\alpha\beta}(r) Y_{lm}^N(\theta, \phi)\hat{\mathbf{e}}_\alpha\hat{\mathbf{e}}_\beta, \tag{6}$$

where $N = \alpha + \beta$. Finally, we also express the fourth-order general viscoelastic tensor $\mathbf{\Lambda}$ in terms of generalized spherical harmonics of angular order s and azimuthal order t:

$$\mathbf{\Lambda}(r, \theta, \phi) = \sum_{s=0}^{\infty}\sum_{t=-s}^{s}\sum_{\alpha,\beta,\gamma,\delta} \Lambda_{st}^{\alpha\beta\gamma\delta}(r) Y_{st}^N(\theta, \phi)\hat{\mathbf{e}}_\alpha\hat{\mathbf{e}}_\beta\hat{\mathbf{e}}_\gamma\hat{\mathbf{e}}_\delta, \tag{7}$$

where we have written $N = (\alpha + \beta + \gamma + \delta)$. α, β, γ and δ can take the values +1, 0 and –1, and t takes integer values between -s and s. l and m are the angular and azimuthal order of the normal mode multiplet and singlet considered; s and t are the angular and azimuthal order of elastic and anelastic Earth structure described by $\mathbf{\Lambda}$. Note that for anisotropy in the Earth, the tensor $\mathbf{\Lambda}$ is restricted to be symmetric:

$$\Lambda_{st}^{\alpha\beta\gamma\delta} = \Lambda_{st}^{\beta\alpha\gamma\delta} = \Lambda_{st}^{\gamma\delta\alpha\beta}. \tag{8}$$

The anisotropy matrix elements then become (Mochizuki 1986; Li *et al.* 1991; Tromp 1995)

$$A_{mm'} = \sum_{s=0,2,4,\dots}\sum_{t=-s}^{s}\sum_{\alpha,\beta,\gamma,\delta,\gamma',\delta'} \int_0^a E^{\alpha\beta *}\Lambda_{st}^{\alpha\beta\gamma\delta}E^{\gamma'\delta'}g_{\gamma\gamma'}g_{\delta\delta'}r^2\mathrm{d}r$$
$$\times \int_\Omega Y_{lm}^{(\alpha+\beta)*}Y_{st}^N Y_{lm'}^{(\gamma'+\delta')}\mathrm{d}\Omega, \tag{9}$$

where a is the radius of the Earth, and $g_{\alpha\alpha'}$ and $g_{\beta\beta'}$ are the elements of the canonical metric tensor for contractions, given in Appendix A, and $N = (\alpha + \beta + \gamma + \delta)$. Using Wigner-3$j$ symbols (Edmonds 1960), the anisotropy matrix can be written as

$$A_{mm'} = \sum_{s=0,2,4,\dots}\sum_{t=-s}^{s}(-1)^m(2l+1)\left(\frac{2s+1}{4\pi}\right)^{1/2}$$
$$\times \begin{pmatrix} l & s & l \\ -m & t & m' \end{pmatrix}\sum_{N=0}^{4}\sum_{i=1}^{i_N}\int_0^a K_{Ni}^s r^2\mathrm{d}r. \tag{10}$$

Only even values of s are allowed in the above because we have made the self-coupling approximation. For self-coupling, N takes integer values between 0 and 4, and i takes integer values between 1 to i_N, where $i_0 = 5$, $i_1 = 3$, $i_2 = 3$, $i_3 = 1$ and $i_4 = 1$, giving a total of 13 radial kernels of the viscoelastic $\mathbf{\Lambda}$ tensor, K_{Ni}^s. The kernels K_{Ni}^s are given by Mochizuki (1986) in their Appendix B (in general format, including cross-coupling and toroidal modes), and by Tromp (1995) (simplified for self-coupling). The kernels depend on $\Lambda_{st}^{\alpha\beta\gamma\delta}$ and the spheroidal mode eigenfunctions U and V. In the case of self-coupling, there are 13 independent contravariant components of the tensor $\Lambda_{st}^{\alpha\beta\gamma\delta}$.

3 TRANSVERSE ISOTROPY WITH CYLINDRICAL SYMMETRY

Here, we consider the effect of transverse isotropy, a simple and symmetric case of the general anisotropy discussed above. This approach has been succesfully applied to elastic anisotropy in the inner core by various authors (e.g. Woodhouse *et al.* 1986; Li *et al.* 1991; Tromp 1995), and extends naturally to the case where both the elastic and the anelastic structures may exhibit anisotropy. To avoid confusion, we note that here the transverse isotropy is defined with respect to the x_3-axis (the Earth's rotation axis, or the north–south axis), that is, to be cylindrically symmetric. This is different from, for instance, PREM (Dziewoński & Anderson 1981), which has a transversely isotropic upper mantle, but the transverse isotropy exhibits radial, not cylindrical, symmetry. We also note that we only consider anisotropy of the Earth's inner core, thus restricting our radial and volume integrations to encompass the inner core only. This also implies that we are only interested in spheroidal normal modes, as toroidal modes with inner core sensitivity (other than those that may be created through leakage of spheroidal energy into toroidal energy as a result of inner core heterogeneity) do not exist.

For a cylindrical symmetry with the x_3-axis (the z-axis) as the symmetry axis, only nine components of the complex viscoelastic tensor $\mathbf{\Lambda}$ are unique and non-zero; in Cartesian coordinates, these components are, in terms of the complex equivalents of the Love parameters (Love 1927)

$$\Lambda^{1111} = \Lambda^{2222} = A^R + iA^I, \Lambda^{3333} = C^R + iC^I,$$
$$\Lambda^{1133} = \Lambda^{2233} = F^R + iF^I, \tag{11}$$
$$\Lambda^{1313} = \Lambda^{2323} = L^R + iL^I, \Lambda^{1212} = N^R + iN^I,$$
$$\Lambda^{1122} = (A^R - 2N^R) + i(A^I - 2N^I). \tag{12}$$

There are only five independent elastic parameters: A^R, C^R, L^R, N^R and F^R; likewise, there are also only five independent anelastic parameters: A^I, C^I, L^I, N^I and F^I. This is an extension of the elastic case, in which only the real Love parameters are considered; the extension to complex parameters in the presence of anelasticity follows naturally by adding imaginary parts A^I to the real elastic

Table 1. Generalized spherical harmonic expansion coefficients $\Lambda_{st}^{\alpha\beta\gamma\delta}$ for the tensor $\mathbf{\Lambda}$, after Tromp (1995). The parameters λ_1, λ_2, λ_3, λ_4 and λ_5 are now complex, with the real parts λ_i^R and imaginary parts λ_i^I.

	$s=0$	$s=2$	$s=4$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{0000}$	$\frac{1}{15}(\lambda_1+2\lambda_2)$	$\frac{4}{21}(\lambda_3+2\lambda_4)$	$\frac{8}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm\pm\mp\mp}$	$\frac{2}{15}\lambda_2$	$-\frac{4}{21}\lambda_4$	$\frac{2}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm\mp\pm\mp}$	$\frac{1}{15}(\lambda_1+\lambda_2)$	$-\frac{2}{21}(\lambda_3+\lambda_4)$	$\frac{2}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm\mp00}$	$-\frac{1}{15}\lambda_1$	$-\frac{1}{21}\lambda_3$	$\frac{4}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm0\mp0}$	$-\frac{1}{15}\lambda_2$	$-\frac{1}{21}\lambda_4$	$\frac{4}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm000}$		$\frac{1}{7\sqrt{3}}(\lambda_3+2\lambda_4)$	$\frac{4}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm\pm\mp0}$		$-\frac{2}{7\sqrt{3}}\lambda_4$	$\frac{2}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm\mp\pm0}$		$-\frac{1}{7\sqrt{3}}(\lambda_3+\lambda_4)$	$\frac{2}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm\pm00}$		$\frac{2}{7\sqrt{6}}\lambda_3$	$\frac{4}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm0\pm0}$		$\frac{2}{7\sqrt{6}}\lambda_4$	$\frac{4}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm\pm\pm\mp}$		$-\frac{2}{7\sqrt{6}}(\lambda_3+2\lambda_4)$	$\frac{2}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm\pm\pm0}$			$\frac{2}{\sqrt{70}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\Lambda_{s0}^{\pm\pm\pm\pm}$			$\frac{4}{\sqrt{70}}\lambda_5$

parameters A^R, and so on. Note that A^R, C^R, L^R, N^R and F^R are called A, C, L, N, F in previous works, in which they are purely real.

To obtain the canonical components $\Lambda_{st}^{\alpha\beta\gamma\delta}$ of the tensor $\mathbf{\Lambda}$, we transform from Cartesian to spherical coordinates, and then use the components listed in Appendix A of Mochizuki (1986). We give the resulting $\Lambda_{st}^{\alpha\beta\gamma\delta}$ tensor components in Table 1, in the same format as Tromp (1995). The complex parameters λ_1, λ_2, λ_3, λ_4 and λ_5 in Table 1 are defined in terms of the complex Love parameters as follows:

$$\lambda_1 = \lambda_1^R + i\lambda_1^I = (6A^R + C^R - 4L^R - 10N^R + 8F^R) + i(6A^I + C^I - 4L^I - 10N^I + 8F^I), \quad (13)$$

$$\lambda_2 = \lambda_2^R + i\lambda_2^I = (A^R + C^R + 6L^R + 5N^R - 2F^R) + i(A^I + C^I + 6L^I + 5N^I - 2F^I), \quad (14)$$

$$\lambda_3 = \lambda_3^R + i\lambda_3^I = (-6A^R + C^R - 4L^R + 14N^R + 5F^R) + i(-6A^I + C^I - 4L^I + 14N^I + 5F^I), \quad (15)$$

$$\lambda_4 = \lambda_4^R + i\lambda_4^I = (A^R + C^R + 3L^R - 7N^R - 2F^R) + i(A^I + C^I + 3L^I - 7N^I - 2F^I), \quad (16)$$

$$\lambda_5 = \lambda_5^R + i\lambda_5^I = (A^R + C^R - 4L^R - 2F^R) + i(A^I + C^I - 4L^I - 2F^I). \quad (17)$$

Now the expansion coefficients given in Table 1 can be used to obtain $\Lambda_{st}^{\alpha\beta\gamma\delta}$. We note that all non-zero expansion coefficients have $t = 0$, that is, only zonal coefficients are included, as is expected for cylindrical symmetry. The non-zero elements are completely determined using terms of angular degree $s = 0$, 2 and 4. There are five non-zero elements with angular degree 0, which depend on the parameters λ_1 and λ_2 only; there are 11 non-zero elements with angular degree 2, which depend on the parameters λ_3 and λ_4 only, and finally, there are 13 non-zero elements with angular degree 4, which depend on the parameter λ_5 only. The parameters λ_1 and λ_2 relate to the isotropic (spherically symmetric) properties, whereas the parameters λ_3, λ_4 and λ_5 completely determine the anisotropy properties. It is useful to recast the normal mode transverse isotropy problem in terms of the parameters λ_3, λ_4 and λ_5 only, as then the degree 2 and 4 contributions can be treated independently. For transverse isotropy in elasticity and anelasticity, then, there are six parameters that fully describe the anisotropy: λ_3^R, λ_4^R, λ_5^R (elastic) and λ_3^I, λ_4^I and λ_5^I (anelastic).

In general terms, the contribution to the splitting matrix $\mathbf{M}$ arising due to cylindrically symmetric anisotropy can be written as

$$A_{mm'} = \sum_{s=0,2,4} \gamma_{s0}^{mm'}\sigma_{s0} = \sum_{s=0,2,4} \gamma_{s0}^{mm'}(c_{s0} + i d_{s0}), \quad (18)$$

where σ_{s0} are the splitting function coefficients; they are divided into c_{s0}, the elastic splitting function coefficients, and d_{s0}, the anelastic splitting function coefficients, and only the zonal ($t = 0$) coefficients are considered here, and

$$\gamma_{st}^{mm'} = (-1)^m(2l+1)\left(\frac{2s+1}{4\pi}\right)^{1/2}\begin{pmatrix} l & s & l \\ -m & t & m' \end{pmatrix}\begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}. \quad (19)$$

Comparing the expression in terms of anisotropy kernels (eq. 10) to $A_{mm'}$ written in terms of splitting function coefficients (eq. 18), we find that the splitting function coefficients arising from cylindrically symmetric anisotropy are given by

$$\sigma_{s0} = \left(\frac{2s+1}{4\pi}\right)^{-1/2} I_s \Big/ \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}, \quad (20)$$

where the integrals I_s are given in terms of the radially integrated kernels K_{Ni}^s of eq. (10) (Mochizuki 1986; Tromp 1995) by

$$I_s = \sum_{N=0}^{s}\sum_{i=1}^{i_N}\left(\frac{2s+1}{4\pi}\right)^{1/2}\int_0^a K_{Ni}^s r^2 dr, \quad (21)$$

where $s = 0$, 2 and 4. The summation is over $i_0 = 5$, $i_1 = 3$, $i_2 = 3$, $i_3 = 1$ and $i_4 = 1$ as before. The anisotropy kernels K_{Ni}^s are given in Appendix B by Mochizuki (1986), repeated by Tromp (1995), and given here again for completeness for the self-coupled case for spheroidal modes considered here in Appendix B. Note that instead of 21 as considered by Mochizuki (1986), there are only 13 independent K_{Ni}^s; this is due to the degeneracy of the $\mathbf{\Lambda}$ elements (Table 1) in the self-coupling case.

Hence, the splitting function coefficients σ_{s0}, for $s = 2,4$, depend on the complex parameters λ_3, λ_4 and λ_5. It is convenient to divide this problem into two independent problems, one for the real parts of σ_{s0} (the elastic splitting function coefficients c_{s0}), and the other for the imaginary parts of σ_{s0} (the anelastic splitting function coefficients d_{s0}). Thus, c_{s0} only depend on λ_3^R, λ_4^R and λ_5^R, and d_{s0} only depend on λ_3^I, λ_4^I and λ_5^I. In simplified form, the two independent equations for the elastic and anelastic splitting function coefficients are then

$$c_{s0} = \int_0^a \lambda_3^R(r)K_{\lambda_3}(r) + \lambda_4^R(r)K_{\lambda_4} + \lambda_5^R(r)K_{\lambda_5}, \mathrm{d}r \quad (22)$$

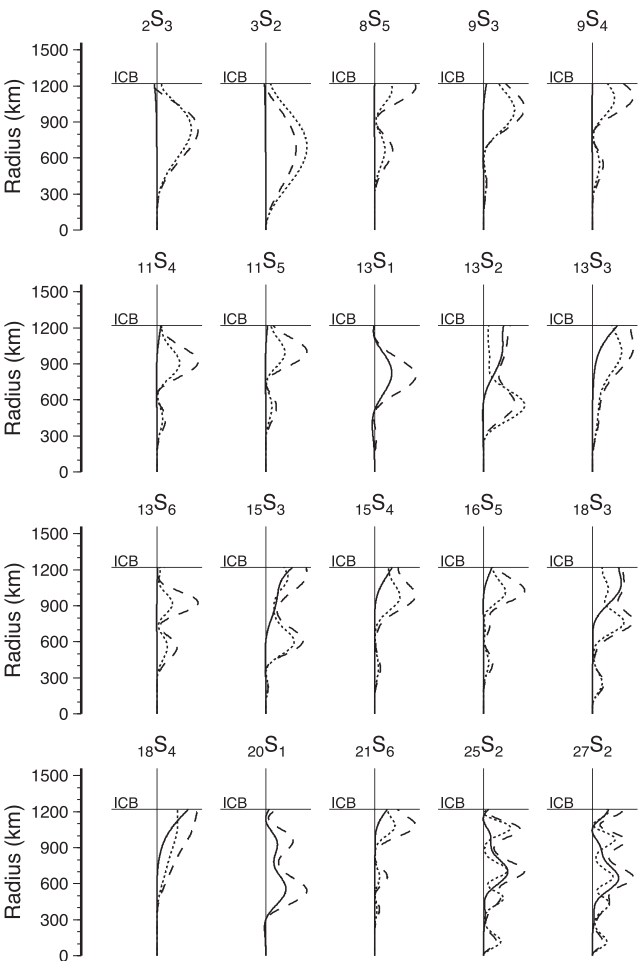

Figure 1. The sensitivity kernels K_{λ_3}, K_{λ_4}, K_{λ_5} for the 20 self-coupled modes whose anelastic splitting function coefficients d_{s0} were measured by Mäkinen & Deuss (2013). ICB stands for the inner core boundary; these kernels only exist within the inner core. K_{λ_3} = solid black line, K_{λ_4} = dashed line, K_{λ_5} = dotted line. Modes $_{13}S_1$ and $_{20}S_1$ have no $s = 4$ coefficients and therefore no λ_5 sensitivity.

and

$$d_{s0} = \int_0^a \lambda_3^I(r) K_{\lambda_3}(r) + \lambda_4^I(r) K_{\lambda_4} + \lambda_5^I(r) K_{\lambda_5} \mathrm{d}r, \tag{23}$$

where the kernels K_{λ_i} are constructed from the integrals I_s (eq. 20) and the kernels K^s_{Ni} (eq. 21) and other coefficients given above, for either $s = 2$ or $s = 4$. Thus the anelastic splitting function coefficients d_{s0} and the elastic splitting functions coefficients c_{s0} depend on the same kernels. These kernels, which only exist within the inner core, are illustrated in Fig. 1 for the 20 inner core sensitive modes whose anelastic splitting function coefficients d_{s0} have been measured recently (Mäkinen & Deuss 2013).

4 RECASTING FOR α, β AND γ, AND DERIVATION OF *P*- AND *S*-WAVE ANISOTROPY

In the previous section we established the framework for calculating elastic c_{s0} and anelastic d_{s0} ($s = 2, 4$) given elastic parameters (λ_3^R, λ_4^R, λ_5^R) and anelastic parameters (λ_3^I, λ_4^I, λ_5^I), respectively. These six parameters and their depth dependence are the natural language in which to discuss velocity and attenuation anisotropy in a normal mode context. However, most velocity anisotropy studies for the inner core have been carried out using body waves, and even those that utilize normal modes (e.g. Woodhouse *et al.* 1986; Tromp 1993; Romanowicz *et al.* 1996; Durek & Romanowicz 1999; Ishii *et al.* 2002; Beghein & Trampert 2003) usually express their results in terms of the parameters α, β and γ, not λ_3^R, λ_4^R, and λ_5^R. To facilitate comparisons between our work and previous studies, we have chosen to do the same; furthermore, we seek to establish how the anelastic λ_3^I, λ_4^I and λ_5^I are to be used for comparisons with body wave studies of attenuation anisotropy in the inner core.

Corresponding to the Love parameters *A*, *C*, *L*, *N* and *F* now being complex, we define α, β and γ to be complex parameters, with the following real and imaginary parts:

$$\alpha^R = \frac{C^R - A^R}{A_0^R}, \quad \alpha^I = \frac{C^I - A^I}{A_0^R}, \tag{24}$$

$$\beta^R = \frac{L^R - N^R}{A_0^R}, \quad \beta^I = \frac{L^I - N^I}{A_0^R}, \tag{25}$$

$$\gamma^R = \frac{A^R - 2N^R - F^R}{A_0^R}, \quad \gamma^I = \frac{A^I - 2N^I - F^I}{A_0^R}, \tag{26}$$

where $A_0^R = \rho_0 V_{p0}^2$ is a constant at the centre of the Earth in the spherical reference model PREM (Dziewoński & Anderson 1981), for (isotropic) *P*-wave velocity V_{p0} and density ρ_0. We note that $A^R = \rho V^2_{p,\mathrm{equatorial}}$, $C^R = \rho V^2_{p,\mathrm{polar}}$, $L^R = \rho V^2_{s,\mathrm{polar}}$, $N^R = \rho V^2_{s,\mathrm{equatorial}}$, where 'polar' and 'equatorial' refer to the wave propagation direction, and the *S*-wave polarization is understood to be in the plane of propagation. Thus $\alpha^R > 0$ signifies polar *P*-wave propagation being faster than equatorial *P*-wave propagation, and $\beta^R > 0$ signifies polar *S*-wave propagation being faster than equatorial *S*-wave propagation. γ^R relates to *P*- and *S*-waves propagating at intermediate angles ξ. The real and imaginary parts of the parameters λ_3, λ_4 and λ_5 can then be written as linear combinations of these parameters:

$$\lambda_3^R = A_0^R[\alpha^R - 4\beta^R - 5\gamma^R], \quad \lambda_3^I = A_0^R[\alpha^I - 4\beta^I - 5\gamma^I], \tag{27}$$

$$\lambda_4^R = A_0^R[\alpha^R + 3\beta^R + 2\gamma^R], \quad \lambda_4^I = A_0^R[\alpha^I + 3\beta^I + 2\gamma^I], \tag{28}$$

$$\lambda_5^R = A_0^R[\alpha^R - 4\beta^R + 2\gamma^R], \quad \lambda_5^I = A_0^R[\alpha^I - 4\beta^I + 2\gamma^I]. \tag{29}$$

We note that α^R, β^R and γ^R correspond to α, β and γ quoted elsewhere. Illustrations of the normal mode sensitivity kernels in terms of these parameters are shown in Fig. 2 for the 20 inner core sensitive modes of Fig. 1. It is interesting that the K_α kernels are much larger than K_β and K_γ, suggesting that the compressional wave anisotropy may be better constrained than the shear wave anisotropy. In deriving the velocity and attenuation anisotropy of P- and S-waves, we follow the treatment of Crampin (1981) (Section 6 for velocity anisotropy and Section 8 for attenuation anisotropy). Again, attenuation is introduced by adding imaginary parts to the Love parameters. We note that the symmetry system used by Crampin (1981) has been adapted to our cylindrically symmetric inner core with the x_3-axis as the symmetry axis, as explained in Appendix C.

The complex parameter $\rho\overline{V}^2$ is given in terms of the elements of the complex tensor $\mathbf{\Lambda}$ by

$$\rho\overline{V}^2 = \hat{\mathbf{k}}\hat{\mathbf{p}} : \mathbf{\Lambda} : \hat{\mathbf{k}}\hat{\mathbf{p}}, \tag{30}$$

where the unit vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{p}}$ give the propagation direction and polarization direction of a body wave, ρ is the density, and $\overline{V}$ is the

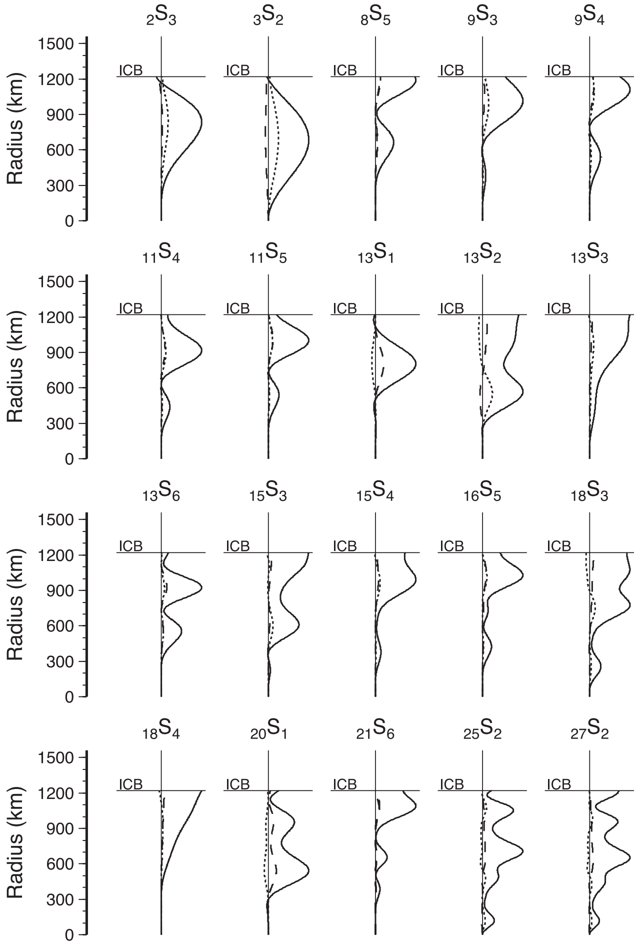

Figure 2. The sensitivity kernels K_α, K_β, K_γ for the 20 self-coupled modes whose anelastic splitting function coefficients d_{s0} were measured by Mäkinen & Deuss (2013). ICB stands for the inner core boundary; these kernels only exist within the inner core. K_α = solid black line, K_β = dashed line, K_γ = dotted line. These kernels are linear combinations of the K_{λ_3}, K_{λ_4}, K_{λ_5} shown in Fig. 1, with the $s = 2$, 4 contributions combined. Note the dominant α sensitivity.

complex velocity. The complex parameter $\rho\overline{V}^2$ can then be divided into real (e^R) and imaginary (e^I) contributions:

$$\rho\overline{V}^2 = e^R + ie^I = \rho V^2 + ie^I, \tag{31}$$

where V is the real body wave velocity. The attenuation coefficient, or quality factor, is defined as

$$\frac{1}{Q} = q = \frac{e^I}{e^R}. \tag{32}$$

Following Crampin (1981), we obtain equations for the complex parameters $\rho\overline{V}^2$ in terms of the complex Love parameters for cylindrical anisotropy (eqs 11 and 12):

$$\rho\overline{V}_p^2 = A - 2(A - F - 2L)\cos^2\xi + (A + C - 2F - 4L)\cos^4\xi, \tag{33}$$

$$\rho\overline{V}_{s_{\rm me}}^2 = L + (A + C - 2F - 4L)(\cos^2\xi - \cos^4\xi), \tag{34}$$

$$\rho\overline{V}_{s_{\rm eq}}^2 = N + (L - N)\cos^2\xi, \tag{35}$$

where $\overline{V}_p$ relates to *P*-waves, $\overline{V}_{s_{\rm me}}$ relates to meriditionally polarized *S*-waves, $\overline{V}_{s_{\rm eq}}$ relates to equatorially polarized *S*-waves, and the Love parameters A, C, L, N and F are complex, that is, $A = A^R + iA^I$ and so on. The angle ξ is now the angle between the direction of wave propagation and the x_3 symmetry axis, which is the Earth's rotation axis in body wave studies of inner core anisotropy. Hence, $\xi = 0$ is a polar path, and $\xi = \pi/2$ is an equatorial path.

To obtain velocity and attenuation anisotropy separately, we rewrite eqs (33)–(35) in terms of the parameters (α^R, β^R, γ^R) and (α^I, β^I, γ^I) of eqs (24)–(26), and then consider the real and imaginary parts separately to obtain e^R and e^I, respectively. For *P*-waves, eq. (33) yields the real and imaginary parts

$$e_p^R = \rho V_p^2 = A^R + A_0^R(4\beta^R - 2\gamma^R)\cos^2\xi + A_0^R(\alpha^R - 4\beta^R + 2\gamma^R)\cos^4\xi, \tag{36}$$

$$e_p^I = A^I + A_0^R(4\beta^I - 2\gamma^I)\cos^2\xi + A_0^R(\alpha^I - 4\beta^I + 2\gamma^I)\cos^4\xi. \tag{37}$$

We start by considering velocity anisotropy using eq. (36). For weak anisotropy, we perturb ρV_p^2 around the isotropic reference value $\rho V_{p,\rm ref}^2$. Using $\rho V_{p,\rm ref}^2 = A^R$, we get

$$\rho V_p^2 = \rho V_{p,\rm ref}^2 + \rho\delta V_p^2 = \rho V_{p,\rm ref}^2 + 2\rho V_{p,\rm ref}\delta V_p = A^R + 2\rho V_{p,\rm ref}^2\frac{\delta V_p}{V_p}. \tag{38}$$

Substituting this into the left-hand side of eq. (36) then gives

$$\frac{\delta V_p}{V_p} = \frac{1}{2}\frac{A_0^R}{A^R}(4\beta^R - 2\gamma^R)\cos^2\xi + \frac{1}{2}\frac{A_0^R}{A^R}(\alpha^R - 4\beta^R + 2\gamma^R)\cos^4\xi. \tag{39}$$

Eq. (39) and its variations have been utilized extensively in both normal mode and body wave studies of velocity anisotropy in the Earth's inner core (e.g. Morelli *et al.* 1986; Shearer *et al.* 1988; Creager 1992; Ishii *et al.* 2002). The total amount of *P*-wave velocity anisotropy is usually defined as the difference between polar ($\xi = 0$) and equatorial ($\xi = \pi/2$) paths, which gives

$$\left(\frac{\delta V_p}{V_p}\right)_{\rm ani} = \frac{1}{2}\frac{A_0^R}{A^R}\alpha^R. \tag{40}$$

Thus, the amount of *P*-wave velocity anisotropy is completely determined by α^R.

We then obtain the attenuation anisotropy using eq. (32) and noting that the velocity anisotropy is weak. Thus, when dividing e_P^I (eq. 37) by e_p^R (eq. 36), all terms in attenuation q_p that depend on α^R, β^R and γ^R can be neglected. This treatment yields

$$q_p = \frac{e^I}{e^R} = \frac{A^I}{A^R} + \frac{A_0^R}{A^R}(4\beta^I - 2\gamma^I)\cos^2\xi + \frac{A_0^R}{A^R}(\alpha^I - 4\beta^I + 2\gamma^I)\cos^4\xi, = q_p + \delta q_p, \tag{41}$$

resulting in the attenuation anisotropy for *P*-waves:

$$\delta q_p = \frac{A_0^R}{A^R}(4\beta^I - 2\gamma^I)\cos^2\xi + \frac{A_0^R}{A^R}(\alpha^I - 4\beta^I + 2\gamma^I)\cos^4\xi. \tag{42}$$

Likewise, for $S_{\rm me}$-waves, we rewrite eq. (34) in terms of (α^R, β^R, γ^R) and (α^I, β^I, γ^I), and taking the real and imaginary parts

$$e^R_{s_{\rm me}} = \rho V^2_{s_{\rm me}} = L^R + A^R_0(\alpha^R - 4\beta^R + 2\gamma^R)(\cos^2\xi - \cos^4\xi), \tag{43}$$

$$e^I_{s_{\rm me}} = L^I + A^R_0(\alpha^I - 4\beta^I + 2\gamma^I)(\cos^2\xi - \cos^4\xi), \tag{44}$$

which yield, following the same treatment as for P-waves,

$$\frac{\delta V_{s_{\rm me}}}{V_{s_{\rm me}}} = \frac{1}{2}\frac{A^R_0}{L^R}(\alpha^R - 4\beta^R + 2\gamma^R)(\cos^2\xi - \cos^4\xi) \tag{45}$$

$$\delta q_{s_{\rm me}} = \frac{A^R_0}{L^R}(\alpha^I - 4\beta^I + 2\gamma^I)(\cos^2\xi - \cos^4\xi). \tag{46}$$

Finally, for $S_{\rm eq}$ -waves, eq. (35) gives the real and imaginary parts

$$e^R_{s_{\rm eq}} = \rho V^2_{s_{\rm eq}} = N^R + A^R_0\beta^R\cos^2\xi, \tag{47}$$

$$e^I_{s_{\rm eq}} = N^I + A^R_0\beta^I\cos^2\xi, \tag{48}$$

which give the velocity and attenuation anisotropy as

$$\frac{\delta V_{s_{\rm eq}}}{V_{s_{\rm eq}}} = \frac{1}{2}\frac{A^R_0}{N^R}\beta^R\cos^2\xi, \tag{49}$$

$$\delta q_{s_{\rm eq}} = \frac{A^R_0}{N^R}\beta^I\cos^2\xi. \tag{50}$$

Hence, α^R, β^R and γ^R completely determine the velocity anisotropy, and the corresponding imaginary parts α^I, β^I and γ^I completely determine the attenuation anisotropy. Again, $\alpha^I > 0$ signifies that P-waves propagating in the polar direction are attenuated more strongly than P-waves propagating in the equatorial direction, and $\beta^I > 0$ signifies S-wave attenuation being stronger in the polar than in the equatorial direction. γ^I relates to attenuation of P- and S-waves propagating at intermediate angles ξ. α^R, β^R and γ^R can be quoted as fractions or percentages, whereas α^I, β^I and γ^I must be divided by a reference compressional attenuation q_p to obtain a percentage value with respect to that reference attenuation. As α^I, β^I and γ^I can be obtained from the anelastic splitting function coefficients d_{s0}, it is now possible to invert anelastic normal mode splitting function data for models of attenuation anisotropy of the Earth's inner core.

5 SPECIAL CASES OF TRANSVERSE ISOTROPY

5.1 Tilted symmetry axis

Certain analyses of inner core elastic anisotropy indicate that the cylindrical symmetry axis of the anisotropy may not be perfectly aligned with the Earth's north–south axis, with Su & Dziewónski (1995) suggesting an axis tilt of up to 10 degrees. In particular, this idea of symmetry axis tilt proved popular in earlier studies of inner core differential rotation (e.g. Song & Richards 1996; Su *et al.* 1996). More recent body wave analyses of inner core elastic anisotropy (Irving & Deuss 2011), however, find no evidence for such an axis tilt, with the best data fit achieved when the symmetry axis of the inner core elastic anisotropy coincides with the north–south axis of the Earth. Tromp (1995) has laid out the theoretical framework for quantifying the impact of a possible axis misalignment on normal modes in the elastic case, showing that such a tilt shall introduce non-zonal elastic splitting function coefficients c_{st} ($t \neq 0$; eqs 57–59 in that paper). As a straightforward extension, if the symmetry axis of the inner core anelastic anisotropy were tilted, non-zonal anelastic splitting function coefficients d_{st} ($t \neq 0$) would also be expected to arise. At present, however, no evidence for either elastic or anelastic symmetry axis misalignment exists.

5.2 Layered anisotropy

Some studies have proposed that the inner core may comprise several distinct layers, or that there may exist an innermost inner core with distinct elastic anisotropy properties (e.g. Ishii & Dziewoński 2002). Furthermore, using normal modes, for example Beghein & Trampert (2003) find a strong variation of inner core elastic anisotropy with depth. Indeed, as long as the cylindrical symmetry holds, depth varying anisotropy is easily accommodated in our formalism, provided the anisotropy parameters λ_3, λ_4 and λ_5 (or α, β and γ) are allowed to vary with depth. As λ^R and λ^I are independent parameters, elastic and anelastic anisotropy may exhibit dissimilar depth dependence. If the possible depth dependence of the anisotropy parameters is not taken into account, the parameters accommodate the radial average values of the elastic and anelastic anisotropy.

6 CONCLUSIONS

We report a method to calculate the anelastic splitting function coefficients d_{s0} ($s = 2, 4$) from models of P-, $S_{\rm me}$- and $S_{\rm eq}$-wave attenuation anisotropy in the Earth's inner core. To do this, we first formulate the anelastic normal mode splitting function coefficients d_{s0} in terms of the parameters λ^I_3, λ^I_4 and λ^I_5, which are the natural parameters for the normal modes problem as they allow $s = 2$ and $s = 4$ coefficients to be treated independently. This approach is equivalent to that used in calculating the elastic splitting function coefficients c_{s0} from models of velocity anisotropy. We then recast the normal mode problem for α^I, β^I and γ^I, which are the anelastic analogues to the commonly considered elastic (velocity) anisotropy parameters α^R, β^R and γ^R. Finally, we derive equations to calculate inner core attenuation anisotropy for P-waves and meriditionally and equatorially polarized S-waves using given values of α^I, β^I and γ^I.

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APPENDIX A: THE CANONICAL BASIS

The canonical basis discussed by Phinney & Burridge (1973) is summarized here. First, the basis vectors of the canonical basis $\hat{\mathbf{e}}_{-1}$, $\hat{\mathbf{e}}_0$ and $\hat{\mathbf{e}}_{+1}$ are given terms of the basis vectors of the spherical basis $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{r}}$ by

$$\hat{\mathbf{e}}_{-1} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\theta}} - i\hat{\boldsymbol{\phi}}), \tag{A1}$$

$$\hat{\mathbf{e}}_0 = \hat{\mathbf{r}}, \tag{A2}$$

$$\hat{\mathbf{e}}_{+1} = -\frac{1}{\sqrt{2}}(\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}}), \tag{A3}$$

The basis vectors are orthonormal:

$$\hat{\mathbf{e}}^*_\alpha \cdot \hat{\mathbf{e}}_\beta = \delta_{\alpha\beta}, \tag{A4}$$

The metric used for tensor contraction is given by $[\mathbf{G}]_{\alpha\beta} = g_{\alpha\beta} = \hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\beta$, so

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{A5}$$

APPENDIX B: ANISOTROPY KERNELS

Anisotropy kernels K^s_{Ni} for self-coupled spheroidal modes in the presence of cylindrically symmetric transverse isotropy (cylindrically symmetric anisotropy) with the x_3-axis as the symmetry axis. These are from Appendix B of the paper by Mochizuki (1986) (where the tensor $\boldsymbol{\Lambda}$ is labelled $\mathbf{C}$), after the format of Tromp (1995). Note that here the tensor $\boldsymbol{\Lambda}$ comprises the complex parameters λ_i; the elements of $\boldsymbol{\Lambda}$ are given in Table 1.

$$K^s_{01} = \dot{U}^2 \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix} \Lambda^{0000}_{s0}, \tag{B1}$$

$$K^s_{02} = 2\Omega^0_l\Omega^2_l\Omega^0_l\Omega^2_l r^{-2} V^2 \begin{pmatrix} l & s & l \\ -2 & 0 & 2 \end{pmatrix} \Lambda^{++--}_{s0}, \tag{B2}$$

$$K^s_{03} = F^2 \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix} \Lambda^{+-+-}_{s0}, \tag{B3}$$

$$K_{04} = -2F\dot{U} \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix} \Lambda^{+-00}_{s0}, \tag{B4}$$

$$K^s_{05} = 2\Omega^0_l\Omega^0_l X^2 \begin{pmatrix} l & s & l \\ -1 & 0 & 1 \end{pmatrix} \Lambda^{+0-0}_{s0}, \tag{B5}$$

$$K^s_{11} = -4\Omega^0_l X\dot{U} \begin{pmatrix} l & s & l \\ -1 & 1 & 0 \end{pmatrix} \Lambda^{+000}_{s0}, \tag{B6}$$

$$K^s_{12} = -4\Omega^0_l\Omega^2_l\Omega^0_l r^{-1} VX \begin{pmatrix} l & s & l \\ -2 & 1 & 1 \end{pmatrix} \Lambda^{++-0}_{s0}, \tag{B7}$$

$$K^s_{13} = 4\Omega^0_l XF \begin{pmatrix} l & s & l \\ 0 & 1 & -1 \end{pmatrix} \Lambda^{+-+0}_{s0}, \tag{B8}$$

$$K^s_{21} = 4\Omega^0_l\Omega^2_l r^{-1} V\dot{U} \begin{pmatrix} l & s & l \\ -2 & 2 & 0 \end{pmatrix} \Lambda^{++00}_{s0}, \tag{B9}$$

$$K^s_{22} = 2\Omega^0_l\Omega^0_l X^2 \begin{pmatrix} l & s & l \\ -1 & 2 & -1 \end{pmatrix} \Lambda^{+0+0}_{s0}, \tag{B10}$$

$$K^s_{23} = -4\Omega^0_l\Omega^2_l r^{-1} VF \begin{pmatrix} l & s & l \\ -2 & 2 & 0 \end{pmatrix} \Lambda^{+++-}_{s0}, \tag{B11}$$

$$K^s_{31} = -4\Omega^0_l\Omega^2_l\Omega^0_l r^{-1} VX \begin{pmatrix} l & s & l \\ -2 & 3 & -1 \end{pmatrix} \Lambda^{+++0}_{s0}, \tag{B12}$$

$$K^s_{41} = 2\Omega^0_l\Omega^2_l\Omega^0_l\Omega^2_l r^{-2} V^2 \begin{pmatrix} l & s & l \\ -2 & 4 & -2 \end{pmatrix} \Lambda^{++++}_{s0}. \tag{B13}$$

Throughout, an overdot denotes differentiation with respect to radius r, and we define F and X in terms of the normal mode eigenfunctions U and V ($W = 0$ for spheroidal modes) of a mode of angular order l as

$$F = r^{-1}[2U - l(l+1)V], \tag{B14}$$

$$X = \dot{V} + r^{-1}(U - V) \tag{B15}$$

and

$$\Omega^N_l = \left[\frac{1}{2}(l+N)(l-N+1)\right]^{1/2}. \tag{B16}$$

APPENDIX C: DERIVATION OF WAVE VELOCITY VARIATIONS FOR CYLINDRICAL SYMMETRY

The complex body wave velocity $\overline{V}$ is given in terms of the elements of the complex tensor $\mathbf{\Lambda}$:

$$\rho\overline{V}^2 = \hat{\mathbf{k}}\hat{\mathbf{p}} : \mathbf{\Lambda} : \hat{\mathbf{k}}\hat{\mathbf{p}}, \tag{C1}$$

where the unit vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{p}}$ give the propagation direction and polarization of body waves, and ρ is the density. To obtain the entities $\rho\overline{V}^2$ for $\overline{V}_p$, $\overline{V}_{s_{\rm me}}$ and $\overline{V}_{s_{\rm eq}}$, we follow the treatment of Crampin (1981), section 6; in order to consider the cylindrical symmetry, parts of this derivation are given explicitly.

Crampin (1981) defines a coordinate system in which x'_3 is a symmetry axis, that is, $x'_3 = 0$ is a symmetry plane. The velocity variations in the $x'_3 = 0$ plane are obtained by rotating the complex tensor $\mathbf{\Lambda}$ [which Crampin (1981) calls $\mathbf{c}$; in the case of Crampin (1981), this tensor is real] about the x'_3-axis. In the coordinate system rotated by an angle θ' about the x'_3-axis, the complex velocities are then

$$\rho\overline{V}^2_p = A' + B_c\cos 2\theta' + B_s\sin 2\theta' + C_c\cos 4\theta' + C_s\sin 4\theta', \tag{C2}$$

$$\rho\overline{V}^2_{SP} = D + E_c\cos 4\theta' + E_s\sin 4\theta', \tag{C3}$$

$$\rho\overline{V}^2_{SR} = F' + G_c\cos 2\theta' + G_s\sin 2\theta', \tag{C4}$$

where SP refers to *S*-waves polarized parallel to the symmetry plane $x'_3 = 0$, and SR refers to *S*-waves polarized perpendicular to the symmetry plane $x'_3 = 0$, and where the constant coefficients are given in terms of the elastic tensor $\mathbf{\Lambda}$ as

$$A' = \frac{1}{8}\left(3\Lambda_{1'1'1'1'} + 3\Lambda_{2'2'2'2'} + 2\Lambda_{1'1'2'2'} + 4\Lambda_{1'2'1'2'}\right), \tag{C5}$$

$$B_c = \frac{1}{2}\left(\Lambda_{1'1'1'1'} - \Lambda_{2'2'2'2'}\right), \tag{C6}$$

$$B_s = \Lambda_{2'1'1'1'} + \Lambda_{1'2'2'2'}, \tag{C7}$$

$$C_c = \frac{1}{8}\left(\Lambda_{1'1'1'1'} + \Lambda_{2'2'2'2'} - 2\Lambda_{1'1'2'2'} - 4\Lambda_{1'2'1'2'}\right), \tag{C8}$$

$$C_s = \frac{1}{2}\left(\Lambda_{2'1'1'1'} - \Lambda_{1'2'2'2'}\right), \tag{C9}$$

$$D = \frac{1}{8}\left(\Lambda_{1'1'1'1'} + \Lambda_{2'2'2'2'} - 2\Lambda_{1'1'2'2'} + 4\Lambda_{1'2'1'2'}\right), \tag{C10}$$

$$E_c = -C_c, \tag{C11}$$

$$E_s = -C_s, \tag{C12}$$

$$F' = \frac{1}{2}\left(\Lambda_{1'3'1'3'} + \Lambda_{2'3'2'3'}\right), \tag{C13}$$

$$G_c = \frac{1}{2}\left(\Lambda_{1'3'1'3'} - \Lambda_{2'3'2'3'}\right), \tag{C14}$$

$$G_s = \Lambda_{2'3'1'3'}, \tag{C15}$$

where the angle θ' is defined as shown in Fig. C1.

If the angle θ' is measured a direction of sagittal symmetry, in which $x'_2 = 0$ is a symmetry plane, the sine terms in eqs (C2), (C3) and (C4) vanish, giving the reduced equations in the coordinate system of Crampin (1981):

$$\rho\overline{V}^2_p = A' + B_c\cos 2\theta' + C_c\cos 4\theta', \tag{C16}$$

$$\rho\overline{V}^2_{s_{\rm me}} = D + E_c\cos 4\theta', \tag{C17}$$

$$\rho\overline{V}^2_{s_{\rm eq}} = F' + G_c\cos 2\theta', \tag{C18}$$

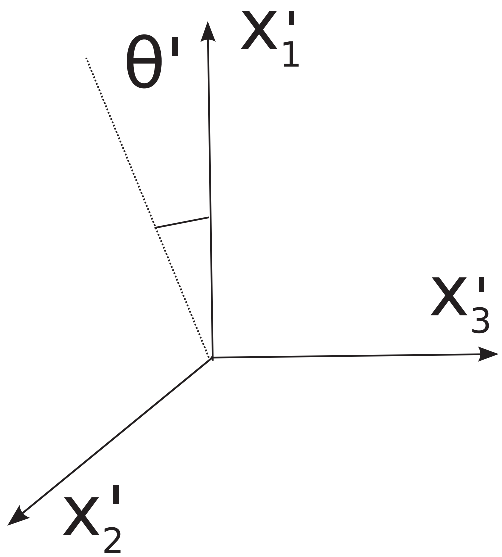

Figure C1. Coordinate system, as used by Crampin (1981), to define symmetry planes for body wave propagation.

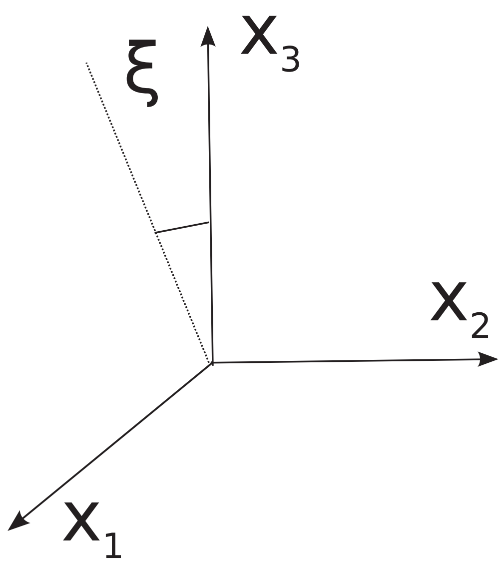

Figure C2. Coordinate system used here to derive the cylindrically symmetric inner core velocity and anelastic anisotropy.

where the symmetry plane $x_3' = 0$ is now the meriditional plane, and the plane perpendicular to it is the equatorial plane. We wish to define a coordinate system in which the x_3-axis (the z-axis) is a symmetry axis, and the angle ξ is the angle from this axis; this coordinate system is shown in Fig. C2.

In order to utilize the work of Crampin (1981), we must then rotate the coordinate system of Fig. C1, and relabel the indices in the following way:

$$1' \rightarrow 3 \tag{C19}$$

$$2' \rightarrow 1 \tag{C20}$$

$$3' \rightarrow 2. \tag{C21}$$

Comparing Figs C1 and C2 reveals the angles θ' and ξ to be equivalent.

We then wish to write the coefficients A', B_c and so on terms of the tensor elements relevant to our cylindrically symmetric coordinate system with x_3 as the symmetry axis. We note that eqs (C16), (C17) and (C18) apply to any symmetry system in which $x_3' = 0$ and $x_2' = 0$ are symmetry planes (or in our coordinate system, $x_2 = 0$ and $x_1 = 0$ are symmetry planes). Such a symmetry system is still more general than our desired cylindrical symmetry system, in which x_3 is a symmetry axis and any plane perpendicular to the $x_3 = 0$ plane is a symmetry plane. In order to consider the desired cylindrical symmetry system, we note that only certain elements of the tensor $\mathbf{\Lambda}$ are non-zero and given in terms of the complex Love parameters.

We then write the coefficients A', B_c and so on in terms of, first the tensor elements Λ_{ijkl}, and then the Love parameters A, C, L, N and F, bearing in mind the general symmetries of the fourth-rank tensor $\mathbf{\Lambda}$:

$$\Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{klij}, \tag{C22}$$

which give

$$\Lambda_{klij} = \Lambda_{lkij} = \Lambda_{ijlk} = \Lambda_{jilk}, \tag{C23}$$

Thus, we obtain the coefficients, now in our coordinate system,

$$A' = \frac{1}{8}(3\Lambda_{3333} + 3\Lambda_{1111} + 2\Lambda_{3311} + 4\Lambda_{3131}) \tag{C24}$$

$$= \frac{1}{8}(3C + 3A + 2F + 4L), \tag{C25}$$

$$B_c = \frac{1}{2}(\Lambda_{3333} - \Lambda_{1111}) \tag{C26}$$

$$= \frac{1}{2}(C - A), \tag{C27}$$

$$C_c = \frac{1}{8}(\Lambda_{3333} + \Lambda_{1111} - 2\Lambda_{3311} - 4\Lambda_{3131}) \tag{C28}$$

$$= \frac{1}{8}(C + A - 2F + 4L), \tag{C29}$$

$$D = \frac{1}{8}(\Lambda_{3333} + \Lambda_{1111} - 2\Lambda_{3311} + 4\Lambda_{3131}) \tag{C30}$$

$$= \frac{1}{8}(C + A - 2F + 4L), \tag{C31}$$

$$E_c = -\frac{1}{8}(\Lambda_{3333} + \Lambda_{1111} - 2\Lambda_{3311} - 4\Lambda_{3131}) \tag{C32}$$

$$= -\frac{1}{8}(C + A - 2F + 4L), \tag{C33}$$

$$F' = \frac{1}{2}(\Lambda_{3232} + \Lambda_{1212}) \tag{C34}$$

$$= \frac{1}{2}(L + N), \tag{C35}$$

$$G_c = \frac{1}{2}(\Lambda_{3232} - \Lambda_{1212}) \tag{C36}$$

$$= \frac{1}{2}(L - N). \tag{C37}$$

Finally, expanding the double and quadruple angles and using these coefficients, now in our cylindrically symmetric system with the angle ξ defined about the symmetry axis x_3, we can write the quantities $\rho\overline{V}^2$ of eqs (C16), (C17) and (C18) as

$$\begin{aligned}\rho\overline{V}_p^2 &= \Lambda_{1111} - 2(\Lambda_{1111} - \Lambda_{1133} - 2\Lambda_{1313})\cos^2\xi \\ &\quad + (\Lambda_{1111} + \Lambda_{3333} - 2\Lambda_{1133} - 4\Lambda_{1313})\cos^4\xi \\ &= A - 2(A - F - 2L)\cos^2\xi + (A + C - 2F - 4L)\cos^4\xi,\end{aligned} \tag{C38}$$

$$\begin{aligned}\rho\overline{V}_{s_{\mathrm{me}}}^2 &= \Lambda_{1313} + (\Lambda_{1111} + \Lambda_{3333} - 2\Lambda_{1133} - 4\Lambda_{1313}) \\ &\quad \times (\cos^2\xi - \cos^4\xi) \\ &= L + (A + C - 2F - 4L)(\cos^2\xi - \cos^4\xi),\end{aligned} \tag{C39}$$

$$\begin{aligned}\rho\overline{V}_{s_{\mathrm{eq}}}^2 &= \Lambda_{1212} + (\Lambda_{2323} - \Lambda_{1212})\cos^2\xi \\ &= N + (L - N)\cos^2\xi.\end{aligned} \tag{C40}$$